Recitation 11

Test tomorrow
Today: Non-independent variables
Test Review
Practice Tests/HW.

Non-independent variables

**Conundrum:** If \( w = x^2 + x^2 + y^2 \) then we can easily compute
\[
\frac{\partial w}{\partial x} = 2x.
\]

However, if we introduce a second equation, \( z = x^2 + y^2 \), we begin to have problems. We have two methods for computing \( \frac{\partial w}{\partial x} \):

**Method A**
Substitute for \( z \):
\[
\omega = x^2 + x^2 + y^2 = \left( x^2 + y^2 \right)^2 + x^2 + y^2 = x^4 + y^4 + 2x^2y^2 + x^2 + y^2
\]
\[
\frac{\partial \omega}{\partial x} = 4x^3 + 4xy^2 + 2x.
\]

**Method B**
Substitute for \( x \):
\[
\omega = z^2 + x^2 + y^2, \quad x^2 = z - y^2
\]
\[
\omega = z^2 + z + y^2 + y^2 = z^2 + z
\]
\[
\frac{\partial \omega}{\partial x} = 0
\]

Why do we get different answers? Which answer is correct?

→ Neither of these methods are "more" or "less" correct than the others. They are different.

Initially, we had 1 eq, \( y \), and → 3 independent vars.
Then added 1 more eq → should have \( \frac{2}{3} \) independence from \( \{ w, x, y, z \} \).
Which ones to choose?

Method (A) chooses \( x, y \) → hold \( y \) const & eval \( \frac{\partial \omega}{\partial x} \)
Method (B) chooses \( x, z \) → hold \( \omega \) const & eval \( \frac{\partial \omega}{\partial x} \)
Notation: $(\frac{\partial w}{\partial x})_y = \text{partial differential of } w \text{ w.r.t. } x\text{ with } y \text{ held constant.}$

$\rightarrow \text{tells you that the designated independent var are } [x, y].$

Example

$w = xy + zx + z^2 \quad z = x+y$

(1) $(\frac{\partial w}{\partial x})_y = ?$

(2) $(\frac{\partial w}{\partial x})_z = ?$

(1) $x, y - \text{ independent, get rid of } z. \rightarrow$

\[ w = xy + (x+y)x + (x+y)^2 = xy + x^2 + xy + x^2 + 2xy + y^2 \]

$= 2x^2 + 4xy + y^2 \rightarrow (\frac{\partial w}{\partial x})_y = 4x + 4y \checkmark$

"Method of substitution"$

(2) \quad x, z - \text{ independent, get rid of } y \quad y = z-x.$

\[ w = x(z-x) + 2z + z^2 \]

$= x^2 + 2zx + z^2 \quad (\frac{\partial w}{\partial z})_x = -2x + 2z \checkmark$

Alternate methods also exist for more complicated cases:
- Chain rule
- Differentials

Chain rule: Apply chain rule to initial formula, holding other variables constant. Take derivative of dependent one, plug in values for these in from the other formulas.

Specified var

Differentials - Take differentials of both equations. May use eq. to get rid of differential of dependent variable.

\[ dw = x \ dy + y \ dx + z \ dz + z \ dx + 2z \ dz \]

\[ = (y+z) \ dx + (x+2z) \ dz + k \ dy \]

\[ dz = dx + dy \]

Keep \( z \) dependent \rightarrow \( dw = (y+z+x+2z) \ dx + (2x+2z) \ dy \)
Kept $z$ dependent $\Rightarrow$ $dw = 4z \, dx + (2x+2z) \, dy$

Kept $y$ dependent $\Rightarrow$ $dw = xdy(y+z) \, dx + (x+2z) \, dz + x(\,dz-dx)$

$$\Rightarrow = (-x+y+z) \, dx + (2x+2z) \, dz.$$ 

How do we use these answers?

Well, just read the partial derivatives off of the coefficients of the $dx$ terms:

$$(x, z \text{-index}) \quad dw = \left( \frac{\partial w}{\partial x} \right) dx + \left( \frac{\partial w}{\partial z} \right) dz$$

so, in $dw = (-x+y+z) \, dx + (2x+2z) \, dz$,

$\frac{\partial w}{\partial x} = -x+y+z = 2z-2x$ \quad \text{from above}$$

$\frac{\partial w}{\partial y} = 2x+2z$ \quad (got this one for free)

Thus, method of differentials computes all partials for specific choice of independent variables.

→ This is a much more general method than what we've discussed.

It is much more straightforward to apply in more complicated examples, esp. ones

→ with > 1 constraint equations
→ with non-hierarchical dependence

Example $w = u^3 - uv^2$

$u = xy$

$v = u + x$

$$\left( \frac{\partial w}{\partial u} \right)_x = ? \quad \left( \frac{\partial w}{\partial x} \right)_u = ?$$

Use differentials $dw = 3u^2 \, du - uv^2 \, dx - 2uv \, dv$ $u$

$du = ydx + xdy$

$dv = du + dx$

$u, v$ - independent $\Rightarrow$ get rid of $dy, dv$.

$dy = du - ydx$ \quad $dv = du + dx$

$-dw = 3u^2 \, du - v^2 \, du - 2uv \, (du+dx) = (6u^2 - 2uv) \, du - 2uv \, dx$

$\left( \frac{\partial w}{\partial u} \right)_x = 3u^2 - v^2 - 2uv$ \quad $\left( \frac{\partial w}{\partial x} \right)_u = -2uv$
HW

\[ g(x, y) = c \quad w = f(x, y, z) \]

\[ \left( \frac{\partial w}{\partial x} \right)_y = \? \quad \left( \frac{\partial w}{\partial x} \right)_z = \? \]

Which of these makes sense?

Hint: Which vars are related by \( g(x, y) = c \)? Which can be dependent?

How to compute partial? → Use method of differentials

Test Review

Concepts
- Gradients
- Directional Derivatives
- Visual analysis of contour lines
- Critical pts & 2nd Derivative test
- Global Extrema
- Lagrange multipliers
- Non-independent variables
- Some PDE questions