Joint Probabilities and Independence

**Joint probability** of 2 events $A$ and $B$ defined in the same sample space (probability that outcome lies in $A$ and $B$):

$$P(AB) = P(C); \text{ where event } C = A \cap B = AB$$

If $A$ and $B$ are **independent** then:

$$P(AB) = P(A)P(B)$$

Note that **mutually exclusive events are not independent** since if one occurs we know the other has not.

Example:

Consider the following events $A$ and $B$ defined from a die toss experiment with outcomes \{1, 2, 3, 4, 5, 6\}

$$A = \{2, 4, 6\} \quad B = \{1, 2, 3, 4\}$$

Then:

$$P(A) = 1/2, \quad P(B) = 2/3, \quad P(AB) = 2/6 = P(A)P(B)$$

So $A$ and $B$ are independent.

**Composite experiments**

Related experiments are often conducted in a sequence.

For example, suppose we toss a fair coin (with 2 equally likely outcomes \{H, T\}) and then throw a fair die (with 6 equally likely outcomes \{1, 2, 3, 4, 5, 6\}). This process can be viewed as two separate experiments $E_1$ and $E_2$ with different sample spaces.

Or ... it can be viewed as a single composite experiment $E$ (with 12 ordered equally likely outcomes \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}).
Events defined in $E_1$ and $E_2$ have equivalent events in $E$.

Example:

$A_2 = \{2, 3\}$ in $E_2$ corresponds to $A = \{H_2, H_3, T_2, T_3\}$ in $E$.

A particular ordered sequence of events from $E_1$ and $E_2$ also has an equivalent event in $E$:

Example:

$A_1 = \{H\}$ in $E_1$ then $A_2 = \{2, 3\}$ in $E_2$ corresponds to $A = \{H_2, H_3\}$ in $E$.

Suppose that $A$ is the composite experiment event that corresponds to event $A_1$ from experiment $E_1$ and then event $A_2$ from experiment in $E_2$.

$A_1$ and $A_2$ are independent if:

$$P_E(A) = P_{E_1}(A_1)P_{E_2}(A_2)$$

The subscript on each probability identifies the corresponding experiment and sample space.

The events $A_1$ and $A_2$ defined in the above coin toss/die roll example satisfy the independence requirement.

Repeated trials

Repeated identical experiments are called repeated trials.

Example:

Consider a composite experiment composed of 3 successive fair coin tosses.

This experiment can yield $2^3 = 8$ equally likely ordered outcomes:

$$\{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

The probability of the event $A = \{\text{exactly 2 heads in 3 tosses}\}$ is the fraction of total number of outcomes that yield exactly 2 heads:

$$P(A) = 3/8$$

Now consider a particular composite experiment event:

$A_1 = \{H\}$ then $A_2 = \{H\}$ then $A_3 = \{T\}$
Suppose the repeated trials are independent. Then the probability of this composite event is:

\[ P( A_1 \text{ then } A_2 \text{ then } A_3 ) = P(A_1)P(A_2)P(A_3) = (1/2)(1/2)(1/2) = 1/8. \]

This is one of 3 mutually exclusive repeated trial event sequences that yield exactly 2 heads. It follows that the probability of exactly 2 heads is \( 3(1/8) = 3/8 \). Since this is equal to the probability obtained from the composite experiment the independence assumption is confirmed.