1) $S_y^2(\omega)$

Note: "+" was inadvertently omitted from assignment.  Okay if you treated it either as $S_y^2(\omega)$ or $S_y(\omega)$ as long as it was treated correctly.

a) VARIANCE = $\sigma_y^2 = \sum_{i=1}^{5} S_y^2(\omega) d\omega$

$\sigma_y^2 = (8)(\frac{1}{2}) + (12)(\frac{1}{2}) + (10)(\frac{1}{2}) + (4)(\frac{1}{2}) + (2)(\frac{1}{2}) = 18 \text{ m}^2$

b) FIND THE AVERAGE UPRCROSSINGS OF THE PLANE $z = 3 \text{ m}$.

$\overline{\eta}(3) = \frac{1}{2\pi} \sqrt{\frac{M_0}{\sigma_y^2}} e^{-\frac{3^2}{2\sigma_y^2}}$

WHERE VARIANCE = $\sigma_y^2 = M_0 = 18 \text{ m}^2$

$M_2 = \sum_{i=1}^{5} \omega_i^2 S_y^2(\omega_i) d\omega_i = (\frac{1}{2})^2(8)(\frac{1}{2}) + (\frac{1}{2})^2(12)(\frac{1}{2}) + (\frac{1}{2})^2(10)(\frac{1}{2})$

$+ (\frac{1}{2})^2(4)(\frac{1}{2}) + (\frac{1}{2})^2(2)(\frac{1}{2}) = 37.5 \text{ m}^2$

$\overline{\eta}(3) = \frac{1}{2\pi} \sqrt{\frac{37.5 \text{ m}^2}{18 \text{ m}^2}} e^{-\frac{9^2}{2(18)}} = 0.1787 \text{ upcrossings/second}$

c) FIND MINIMUM DECK CLEARANCE TO BE FLOODED ONCE PER HOUR.

From part b) $M_0 = 18 \text{ m}^2$, $M_2 = 37.5 \text{ m}^2$

$\overline{\eta}(h) \leq 1 \text{ upcrossing per hour} = \frac{1 \text{ upcrossing}}{3600 \text{ seconds}}$

$\overline{\eta}(h) \leq 0.000278 \text{s}^{-1}$

$\overline{\eta}(h) = \frac{1}{2\pi} \sqrt{\frac{37.5}{18.0}} e^{-\frac{h^2}{2.18}} \leq 0.000278$

$e^{-\frac{h^2}{2.18}} \leq 0.601209$

SIMPLIFIES TO:

$h \geq 15.55 \text{ m}$
2) Given: \( \zeta_0^2 = 18 \text{ m}^2 \), \( \zeta = 0.6 \)

a) With a sea spectrum bandwidth of 0.6, use the approximation:

\[
P(\eta \geq \eta_0) \approx \frac{2 \sqrt{1-e^{-\eta_0^2}}}{1+\sqrt{1-e^{-\eta_0^2}}} e^{-\eta_0^2/2}
\]

\[
P(\eta \geq \eta_0) \approx \frac{2 \sqrt{1-0.6^2}}{1+\sqrt{1-0.6^2}} e^{-0.6^2/2}
\]

\[
P(\eta \geq \eta_0) \approx 0.8888 e^{-0.6^2/2}
\]

To find \( P(\eta \geq 5 \text{ m}) \), \( A = 5 \text{ m} \), and \( \eta_0 \) is a nondimensionalized number.

\[
\eta_0 = \frac{A}{\sqrt{\zeta_0^2}} = \frac{5 \text{ m}}{\sqrt{18 \text{ m}^2}} = 1.17851
\]

\[
\therefore P(\eta \geq 1.17851) \approx 0.8888 e^{-\left(1.17851\right)^2/2}
\]

\[
\boxed{P(\text{wave maxima exceeding 5 m}) \approx 44.4\%}
\]

b) 10 m?

\[
\zeta_0 = \frac{10 \text{ m}}{\sqrt{18 \text{ m}^2}} = 2.357
\]

\[
\therefore P(\eta \geq 2.357) \approx 0.8888 e^{-\left(2.357\right)^2/2}
\]

\[
\boxed{P(\text{wave maxima exceeding 10 m}) \approx 85\%}
\]

c) Find the required deck height to have 1% chance of flooding:

\[
0.01 \approx 0.8888 e^{-\eta_0^2/2}
\]

\[
0.01125 \approx e^{-\eta_0^2/2}
\]

\[
2(\ln 0.01125) \approx -\eta_0^2
\]

\[
\eta_0^2 = 2.99579 = \frac{A}{\sqrt{\zeta_0^2}} \quad \therefore A \approx (2.99579)\sqrt{18 \text{ m}^2}
\]

\[
\boxed{A \approx 17.71 \text{ m}}
\]
3) \[ f(t) = \sum_{i=1}^{N} \xi_i \cos(\omega_i t + \phi_i) \]

**Gaussian with zero mean**

a) \[(m + a_{33}) \ddot{x}(t) + (b_{33}) \dot{x}(t) + (c_{33}) x(t) = f(t)\]

where 
- \( m \) = ship's mass
- \( a_{33} \) = added mass coefficient
- \( b_{33} \) = damping coefficient
- \( c_{33} \) = restoring coefficient
- \( x(t) \) = heave motion

b) \[ H(\omega) = \frac{1}{\sigma^2} e^{i\phi} \]

where \( \sigma^2 = \omega^2 (m + a_{33}) + i \omega b_{33} + c_{33} \)

**c) Input to LTI system is Gaussian with zero mean, therefore output is Gaussian with zero mean**

\[ f(t) \rightarrow \text{LTI} \rightarrow x(t) \]

d) **Variance of the heave is the same as the 0th moment of the spectrum of the heave**

\[ \sigma_x^2 = \int S_x(\omega) d\omega \]

And from Wiener-Khintchine, \( S_x(\omega) = S_f(\omega) |H(\omega)|^2 \)

So \( \sigma_x^2 = \int S_f(\omega) |H(\omega)|^2 d\omega \) (where \( H(\omega) \) is in part b).

e) **Think of acceleration of heave as an LTI system with heave**

\[ x(t) \rightarrow \text{LTI} \rightarrow \ddot{x}(t) \]

From part c), heave is Gaussian with zero mean,
- heave acceleration, \( \ddot{x}(t) \), is also Gaussian with zero mean.
3) (f) \[ f(t) \xrightarrow{\text{LTI}} x(t) \xrightarrow{\text{LTI}} \tilde{x}(t) \]

\[ S_x(\omega) = S_f(\omega) |H(\omega)|^2 \quad S_{\tilde{x}}(\omega) = S_x(\omega) |H_1(\omega)|^2 \]

\[ S_{\tilde{x}}(\omega) = S_f(\omega) |H(\omega)|^2 |H_1(\omega)|^2 \]

where \[ H(\omega) = \frac{1}{-\omega^2(\alpha_x a_x) + i\omega b_x + c_x} e^{i\phi} \]

and \[ H_1(\omega) = -\omega^2 \]

g) Yes, the output from an LTI system with a Gaussian input is also Gaussian.