Chapter 8

Financial Markets, Savings, and Growth

8.1 A Simple $AK$ Model with Incomplete Markets

8.1.1 Model Setup

- A simple $AK$ endogenous-growth model with incomplete markets and uninsured idiosyncratic risk, with or without aggregate uncertainty.

- A continuum of entrepreneurs/agents, $i \in [0, 1]$.

- Each period $t$, entrepreneur $i$ has access to two technologies:
  
  - A common ‘subsistence’ or ‘storage’ technology, which is riskless,

$$G(k) = Rk, \quad R > 0$$
(ii) An $AK$-type technology with individual- or project-specific risk,

\[ f^i_t(k) = A^i_t k \]

where $A^i_t$ is an idiosyncratic productivity shock. c.d.f. $F$ and support $\mathbb{A} = \{ A \in \mathbb{R} | F(A) \geq 0 \} \subseteq \mathbb{R}_+$, $F(0) = 0$.

- $A^i_t$ is i.i.d. across $i$ and $t$, with c.d.f. $F$ over $\mathbb{R}_+$,

\[ \overline{A} \equiv E A^i_{t+1} = E_t [A^i_{t+1}] , \quad \overline{A} > R > \inf \{ A : F(A) > 0 \} . \]

W.l.o.g., $R \geq 1/\beta > 1$.

- Parametrize distribution of $A^i_t$ by $\sigma$:

\[ A^i_t = \overline{A} \exp \{ \sigma \varepsilon^i_t \} \]

$\varepsilon^i_t$ is log-normal.

- Infinite horizon, Epstein-Zin preferences:

\[ u_t = U(c_t) + \beta \cdot U^V \left( \mathbb{E}_t \left[ V^{-1}(u_{t+1}) \right] \right) \]

- CEIS/CRRA preferences:

\[ U(c) = \frac{c^{1/\theta} - 1}{1 - 1/\theta} \]
\[ V(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma} \]

$\gamma$ coefficient of relative risk aversion; $\theta$ elasticity of intertemporal substitution.
8.1.2 Optimal Individual Behavior

- To simplify, the entrepreneur has to fully specialize in one technology: a discrete employment (or portfolio) choice \( l_t \in \{0, 1\} \).

- The budget constraints:
  \[ c^i_t + k^i_t \leq y_t, \quad l_t \in \{0, 1\}, \]
  where
  \[ y_t = l_{t-1}^i A_t^i k_{t-1}^i + (1 - l_{t-1}^i) R k_{t-1}^i = [R + l_{t-1}(A_t^i - R)] k_{t-1}^i \]

- Optimal specialization \( l_t^i \):
  \[ l_t^i = \text{arg max}_{l_t^i \in \{0, 1\}} V^{-1} \left( E_t[V(c_{t+1}^i)] \right) \]

- Solution independent of \( i \) and \( t \) as long as \( A_{t+1}^i \) is i.i.d. across \( i \) and \( t \):
  \[ l_t^i = l \]
  \[ c_t^i = (1 - s)y_t^i \]
  \[ k_t^i = sy_t^i = s \left[ R + l(A_t^i - R) \right] k_{t-1}^i \]

for some constant \( l \in \{0, 1\} \) and \( s \in (0, 1) \).
Optimal $s$ and $l$ such that

$$
s = \beta^\theta \left( \left\{ E_t[R + l(A_{t+1} - R)]^{1-\gamma} \right\}^{\frac{1}{1-\gamma}} \right)^{\theta-1}
$$

$$
l = \arg \max_{l \in \{0,1\}} \left\{ l \cdot [E_t[A_{t+1}^{1-\gamma}]]^{\frac{1}{1-\gamma}} + (1-l') \cdot R \right\}
$$

Define $B$ as the certainty equivalent of the return to the risky technology (the risk-adjusted return):

$$
B \equiv [E[A^{1-\gamma}]]^{\frac{1}{1-\gamma}} = [E_t[(A_t^{1-\gamma})]]^{\frac{1}{1-\gamma}} \equiv B(\sigma)
$$

Note that $B$ decreases with $\sigma$,

$$
\frac{\partial B(\sigma)}{\partial \sigma} < 0
$$

and satisfies

$$
B(0) = \bar{A} > R > 0 = B(\infty)
$$

Thus there is a unique $\tilde{\sigma} \in (0, \infty)$ such that

$$
B(\tilde{\sigma}) = R.
$$

For a risk-free bond,

interest rate $= \max\{B, R\}$

$l^*$ maximizes the return to savings:

$$
B < R \Rightarrow l^* = 0
$$

$$
B > R \Rightarrow l^* = 1
$$
The equilibrium saving rate is then

\[ s^* = \beta^\theta (\text{return to savings})^{\theta-1} \]

where

\[ \text{return to savings} = \max\{B, R\} \]

\( \theta \) is the elasticity of intertemporal substitution

- If high idiosyncratic risk (sufficiently incomplete markets):

  \[ \sigma > \tilde{\sigma} \Rightarrow B < R \]
  \[ \Rightarrow l^* = 0 \Rightarrow s^* = \beta^\theta R^{\theta-1} \]

If low idiosyncratic risk (relatively complete markets):

\[ \sigma < \tilde{\sigma} \Rightarrow B > R \]
\[ \Rightarrow l^* = 1 \Rightarrow s^* = \beta^\theta B^{\theta-1} \]

- Risk, specialization, and savings:

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<thead>
<tr>
<th>( \sigma &gt; \tilde{\sigma} )</th>
<th>( \theta &lt; 1 )</th>
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- For \( \sigma < \tilde{\sigma} \):
  - the risk-adjusted return \( B = [EA^{1-\rho}]^{1/(1-\rho)} \) always falls with risk \( \sigma \);
  - the saving rate \( s^* = \beta^\theta B^{\theta-1} \) increases as \( B \) falls iff \( \theta < 1 \);
  - therefore, the saving rates increases with risk \( \sigma \) iff \( \theta < 1 \).
• Conditional on $l^* = 1$, the savings rate $s^*$ decreases as we complete the markets iff the precautionary-savings effect is strong enough. But if the elasticity of intertemporal substitution is sufficiently high, then completing the markets raises the saving rate as it raises the risk-adjusted real return.

• Remark: If we introduce a riskless bond in zero net supply, the bond market will clear at

$$\text{interest rate} = \max\{B, R\}.$$ 

8.1.3 Aggregates

• For the individual,

$$g_{t+1}^i = \frac{y_{t+1}^i}{y_t^i} = s[R + l(A_{t+1}^i - R)].$$

If $\sigma > \tilde{\sigma}$, $l = 0$, and $g_{t+1}^i = sR$ (non-random)

If $\sigma < \tilde{\sigma}$, $l = 0$, and $g_{t+1}^i = sA_{t+1}^i$ (random).

• For the aggregates,

$$C_t = (1 - s)Y_t, \quad K_t = sY_t.$$ 

If $\sigma > \tilde{\sigma}$,

$$Y_t = RK_{t-1}$$

$$g^* = s^*R = (\beta R)^\theta$$

If instead $\sigma < \tilde{\sigma}$, since idiosyncratic shocks wash out at the aggregate,

$$Y_t = \int_i (A_i^i k_{t-1}^i) = \overline{A}K_{t-1}$$

$$g^* = s^*\overline{A} = \beta^\theta B^{\theta-1}\overline{A}$$

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Hence, aggregates are always deterministic.

- **Aggregate technology:**

\[
Y_k = \begin{cases} R & \iff l^* = 0 \iff B < R \iff \sigma > \tilde{\sigma} \\ A & \iff l^* = 1 \iff B > R \iff \sigma < \tilde{\sigma} \end{cases}
\]

### 8.1.4 Aggregate Growth

- Let \( g^* = (\beta A)^\theta \); this is the complete-markets or first-best growth rate.

- Given that \( A > R \) by assumption, and that \( B < A \) for any \( \sigma > 0 \), we have:

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- Also, for \( \sigma < \tilde{\sigma} \):

\[
\frac{\partial g^*}{\partial \sigma} \text{ same signs as } \frac{\partial s^*}{\partial \sigma}, \text{ same signs as } 1 - \theta
\]

- If the EIS is high, completing the markets increases savings and growth unambiguously.

- When \( l^* = 1 \) and thus \( g^* = \beta B^{\theta-1} A \). This is **not** the growth rate \( g^o = (\beta A)^\theta \) that we would calculate from a representative agent model with technology \( Y = AK \); nor the growth rate \( g = (\beta B)^\theta \) that we would calculate from a representative agent model with technology \( Y = BK \). In particular, \( (\beta B)^\theta < g^* \leq (\beta B)^\theta \). Difference due to market incompleteness. Similarly, interest rate \( B < A \), and \( s^* = \beta B^{\theta-1} \neq \beta A^{\theta-1} = s^o \).

**Proposition 28** For any \( A > R \), there is \( \tilde{\sigma} = \tilde{\sigma}(A, R, \rho) > 0 \) with \( \partial \tilde{\sigma} / \partial A > 0 \), such that

\[
\begin{align*}
\sigma > \tilde{\sigma} & \Rightarrow \begin{cases} l^* = 0, \ s^* = \beta R^{\theta-1} \leq s^o \\ g^* = (\beta R)^\theta < g^o \end{cases} \\
\sigma < \tilde{\sigma} & \Rightarrow \begin{cases} l^* = 1, \ s^* = \beta B^{\theta-1} \leq \beta R^{\theta-1} \\ g^* = \beta B^{\theta-1} A > (\beta R)^\theta, \ g^* \leq g^o \end{cases}
\end{align*}
\]
Show Figure 1.

- The competitive equilibrium is not first-best. However, it is constrained Pareto efficient!

8.1.5 Comparison: Complete Markets vs. Financial Autarchy.

- Assume access to a complete assets market; **fully insure** against all idiosyncratic risk ⇒ a net-of-hedging safe return $\bar{A}$.

- Since $\bar{A} > R$, specialization $l^i_t = 1 \forall t, i$.

- The *representative-agent model* applies and the Euler condition writes

  $$U'(c^i_t) = \beta \bar{A} U'(c^i_{t+1})$$

- *The Arrow-Debreu equilibrium*: For all $i, t$ it holds that

  $$y^i_t = \bar{A} k^i_t, \quad k^i_t = sy^i_t, \quad c^i_t = (1-s)y^i_t$$

  $$g^i_t = s\bar{A} = (\beta \bar{A})^\theta, \quad s = \beta^{\theta-1}$$

- We can thus summarize:

**Proposition 29** If intertemporal substitution is strong ($\theta > 1$), then both the growth rate and the savings rate are higher under complete markets than under financial autarchy. If instead risk intertemporal substitution is weak ($\theta < 1$), then the savings rate is lower under complete markets, and the growth rate may be either higher or lower. If idiosyncratic risk had been sufficiently high (so that $B < R$), then completing the markets unambiguously raises the growth rate, whatever $\theta$. But if idiosyncratic risk had been rather small (so that $B > R$),
and intertemporal substituiton weak ($\theta < 1$), then and only then completing the markets can slow down growth. Finally, the interest rate is unambiguously increasing with market completeness.

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- **Stage I:** Highly incomplete markets, too much uninsurable idiosyncratic risk, $\sigma > \bar{\sigma}$. In this stage, $B < R < \overline{A}$ and $l = 0$.

- **Stage II:** Moderately incomplete markets, sufficiently low uninsurable idiosyncratic risk, $0 < \sigma < \bar{\sigma}$. In this intermediate stage, $\overline{A} > B > R$ and $l = 1$.

- **Stage III:** Complete financial markets, fully insured idiosyncratic risk, $\sigma \approx 0$. In this final stage, the Arrow-Debreu equilibrium applies, $B \approx \overline{A}$ and $l = 1$.

- Empirical implications? Cross-country interpretation? Time-series interpretation?

8.1.7 Growth and Income Distribution: a Kuznets Curve.

- Stage I: low growth and low income dispersion, for nobody takes risks.

- Stage II: output levels and growth rates unambiguously increase, but income dispersion raises as well, for entrepreneurs now take significant uninsurable idiosyncratic risk.
Stage III, more and more of the idiosyncratic risk is insured away, and thus income dispersion falls, due to sufficient risk-sharing.

A inverted-U shaped relation b/w income inequality and market sophistication ⇒ a Kuznets curve.

8.1.8 Progressive Taxation and Social Security as Insurance.

A rational for progressive taxation, or social security: provide insurance, effectively substitute for missing markets.

Progressive taxation may enhance growth if markets are incomplete.

The optimal tax schedule w/o aggregate uncertainty.

Let $T^i_t(.)$ be tax payments individual $i$ makes at $t$.

To implement the Arrow-Debreu allocation after taxes,

$$u'(c^i_t) = \beta E \left[ A^i_{t+1} - \frac{\partial T^i_t(.)}{\partial k^i_t} u'(c^i_{t+1}) \right]$$

$$u'(c^i_t) = \beta \bar{A} u'(c^i_{t+1})$$

Optimal taxation is

$$T^i_t(.) = [A^i_t - \bar{A}]k^i_{t-1} = y^i_t - \frac{Y_t}{K_t} k^i_{t-1}$$

Ensures a certain income level $\bar{A}k^i_t$ and a certain capital return $[A^i_{t+1} - \frac{\partial T^i_t(.)}{\partial k^i_t}] = \bar{A}$ in all states.
The optimal tax schedule in the presence of aggregate fluctuations.

- Allow for exogenous aggregate fluctuations $\tilde{A}_t$.
  
  $A^i_t = \tilde{A}_t + \varepsilon^i_t$; $\varepsilon^i_t$ i.i.d. and independent of $\tilde{A}_t$.
  
  $\tilde{A}_t$ a stationary process bounded from below by $R$.

- The stochastic optimal tax system:
  
  $$T^i_t(.) = [A^i_t - \tilde{A}_t]k^i_{t-1} = y^i_t - \frac{Y_t}{K_t}k^i_{t-1}$$

- Countercyclical taxes:
  
  $$Corr_{t-1}(T^i_t, Y_t) = Corr_{t-1}(T^i_t, \tilde{A}_t) = -1 < 0$$

BUT:

- The above tax implications presume government can observe idiosyncratic shocks $A^i_t$.

- Why should the government be able to do so, and the market not?

- What is the shocks are private information to the agents?