14.102 Midterm Exam
October 17, 2002.

Instructions: This is a closed book exam. Please answer all questions. You have 1 hour 20 minutes to complete the exam. Good luck!

1. (Short questions).

(a) Define the rank of a matrix. What is a nonsingular matrix? Suppose matrices $A$ and $B$ are of rank 1 each. What can be the rank of $A+B$?

(b) Define the equivalence relation. Is the relation “$x \sim y$ iff $d(x, y) \leq 1$” an equivalence relation in $\mathbb{R}^n$?

(c) State the Weierstraß theorem. Is the following statement true: “A monotone (not necessarily continuous) function $f : X \subset \mathbb{R} \to \mathbb{R}$ on a compact set always reaches its maximum and minimum values on $X$”?

(d) State the separation hyperplane theorem. Is the following statement true: “Two (not necessarily convex) sets in $\mathbb{R}^n$ can be separated by a hyperplane as long as their convex hulls have empty intersection”?

(e) Define what it means for a function to be differentiable at a point in $\mathbb{R}^n$. What is the gradient of a function?

2. (Matrix Algebra) Let

$$A = \begin{pmatrix}
5 & 4 & 0 & 0 \\
4 & 5 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}$$

(a) Find $\det A$

(b) Find $\text{rank} A$

(c) Is $A$ positive (negative) definite?

(d) Is $A$ positive (negative) semidefinite?

(e) Find the eigenvalues of $A$.

(f) Find the eigenvectors of $A$. Are they orthogonal?

(g) Under what restrictions (if any) on vector $b \in \mathbb{R}^4$ will the system $Ax = b$ have solutions?

3. (Optimization in $\mathbb{R}^n$) Let $F(x, y) = x^2 + y^2 - 4x - 2y$ and $G(x, y) = x^2 + 2y^2 - 4x - 4y$. 

(a) State the implicit function theorem. Find all points on the curve $G(x, y) = 0$ around which either $y$ is not expressible as a function of $x$ or $x$ is not expressible as a function of $y$. Compute $y'(x)$ along the curve at the origin.

(b) Find all unconstrained optima of $F$ and $G$ on $\mathbb{R}^2$. Is the Weierstraß theorem applicable?

(c) Maximize and minimize $F(x, y)$ subject to $G(x, y) = 0$. Is the Weierstraß theorem applicable?

(d) Maximize and minimize $F(x, y)$ subject to $G(x, y) \leq 0$. Is the Weierstraß theorem applicable?

(e) Maximize and minimize $F(x, y)$ subject to $G(x, y) \geq 0$. Is the Weierstraß theorem applicable?