14.102 Midterm Exam
October 17, 2003.

Instructions: This is a closed book exam. Please answer all questions. You have 2 hours to complete the exam. Good luck!

1. (Short questions, 5 pts each). For true/false questions you should either prove the statement or provide a counterexample.

(a) List the properties of a distance function. Is the following statement true: "if $d(\cdot, \cdot)$ is a distance, then $d'(x, y) = (d(x, y))^2$ is a distance"?

(b) Define a limit point of a sequence. Is it true that if $A$ is a limit point of a sequence $\{a_n\}$ and $B$ is a limit point of a sequence $\{b_n\}$ then $A + B$ is a limit point of sequence $\{a_n + b_n\}$?

(c) Define a symmetric matrix. Is it true that the product of two symmetric matrices is a symmetric matrix?

(d) State the Weierstraß theorem. Is it true that any function that is differentiable on a compact set is bounded?

(e) State the Separating Hyperplane Theorem. Is it true that for any two disjoint closed convex sets $C_1$ and $C_2$ there exists a hyperplane $H(p, a)$ such that $p \cdot x < a$ for all $x \in C_1$ and $p \cdot y > a$ for any $y \in C_2$?

(f) State the implicit function theorem. Find all points on the curve $x^4 - 2x^2y^2 + y^4 = 0$ around which either $y$ is not expressible as a function of $x$ or $x$ is not expressible as a function of $y$. Compute $y'(x)$ along the curve at point $(1, -1)$.

(g) State Kuhn-Tucker Theorem under convexity. What is Slater’s condition?

2. (Matrix Algebra, 25 pts)

Let $A = \begin{pmatrix} 6 & 3 & 0 & 0 \\ 3 & 6 & 0 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$.

(a) Find $\det A$ and $\text{rank} A$

(b) Is $A$ positive/negative definite? Is $A$ positive/negative semidefinite?

(c) Find the eigenvalues of $A$.

(d) Find the eigenvectors of $A$. How many of them are there? Are they orthogonal?

(e) Under what restrictions (if any) on vector $b \in \mathbb{R}^4$ will the system $Ax = b$ have solutions?
3. (Optimization in $\mathbb{R}^n$, 10 pts for each of (a)-(d) and 20 pts for (e)) Let $F(x, y) = \frac{1}{x+y+1}$, $G_p(x, y) = 2x^2 - 4px + 2y^2 + 2y^2 - 18$, where $p$ is a parameter. For questions (a) to (d) assume $p = 1$.

(a) Find all unconstrained optima of $F$ and $G$ on $\mathbb{R}^2$. Is the Weierstraß Theorem applicable?

(b) Maximize and minimize $F(x, y)$ subject to $G_1(x, y) = 0$. Is the Weierstraß Theorem applicable?

(c) Maximize and minimize $F(x, y)$ subject to $G_1(x, y) \leq 0$. Is the Weierstraß Theorem applicable?

(d) Maximize and minimize $F(x, y)$ subject to $G_1(x, y) \geq 0$. Is the Weierstraß Theorem applicable?

(e) (Bonus question, attempt it only if you have extra time) The problem is to maximize $F(x, y)$ subject to $G_p(x, y) \geq 0$. State the Maximum Theorem. Are the hypotheses of it satisfied? Compute directly the function $f^*(p) = \max F(x, y)$ s.t. $G_p(x, y) \geq 0$; is it continuous? Compute $(x^*(p), y^*(p))$ (defined as the set of maximizers of $F(x, y)$ subject to $G_p(x, y) = 0$). Is it a use and/or lsc correspondence? Is the Maximum Theorem under convexity applicable?

4. (Red Sox game may be your friend, 0 pts; do not attempt this problem until after you leave the exam room). Assume that your performance on this exam is given by $p = k + t$, where $k$ is your knowledge and $t$ is how much time you spent preparing last night. I would like to give you the score equal to your knowledge $k$ but it is not observable to me – all I see is your performance $p$. I know however, that your disutility of spending time $t$ preparing for the exam is $D(t) = \frac{t^2}{2}$, and your total utility is your score less disutility of effort: $U_{you}(s, t) = s - D(t)$. My utility of scoring your test at $s$ when your actual knowledge level is $k$ is $U_{me} = -(s-k)^2$. I estimate the time you spent preparing at $t^*$ and give you score $s = p - t^*$.

(a) Find the first best level of preparation $t^{FB}$.

(b) Find the level of preparation $t$ that you will choose given my grading policy $s = p - t^*$.

(c) Find the optimal (for me) grading policy $t^*$ and equilibrium utility levels $U_{you}$ and $U_{me}$.

(d) Suppose that it is a common knowledge that you could not possibly spend more than $t = \frac{1}{4}$ preparing last night (because of the Red Sox game, which was supposed to be 9 innings long). Redo points (b) and (c) under this assumption. Do I have higher or lower utility $U_{me}$? Do you have higher or lower utility $U_{you}$? Is the outcome more efficient or less efficient than that without the game?

(e) Discuss verbally efficiency implications of innings 10 and 11 of the game.