1. Lecture Notes Exercise 105: Given an \( m \times n \) matrix \( A \), show that \( S(B) \subseteq S(A) \) and \( N(A') \subseteq N(B') \) whenever \( B = AX \) for some matrix \( X \). What is the geometric interpretation? (Note: this is a repeat from last year’s problem set; as such, the solution is right on the website. It is certainly worth doing, but the main reason I included it was to draw your attention to the result, which can be used to make part (e) of the next problem much less tedious.)

2. Let \( A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \end{pmatrix} \) and \( B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{pmatrix} \)

(a) Find \( C = AB \)

(b) Find \( \text{rank } C \)

(c) Find \( \det C \)

(d) Find \( D = BA \)

(e) Find \( \text{rank } D \) (reminder: try to answer this using the result of problem 1 - without calculations)

(f) Find \( \det D \)

(g) Is \( C \) invertible? If so, find \( C^{-1} \)

(h) Is \( D \) invertible? If so, find \( D^{-1} \)

(i) Find eigenvalues of \( C \)

(j) Solve the following two linear systems (Hint: you will need no extra calculations!):

i. \( \begin{cases} \frac{1}{2}x + \frac{6}{7}y = 1 \\ \frac{1}{2}x - \frac{1}{7}y = 0 \end{cases} \)

ii. \( \begin{cases} \frac{1}{2}u + \frac{6}{7}v = 0 \\ \frac{1}{2}u - \frac{1}{7}v = 1 \end{cases} \)

3. Look at last year’s problem set 1, #3, and its solution. It is good to understand the notion of changing bases, and of the coordinates of a vector with respect to a basis (we will use it again in discussing diagonalization). In particular, do Lecture Notes Exercise 114: what are the coordinates of \( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) with respect to the following bases?

(a) \( \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \)

(b) \( \begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix} \)
4. Prove Lemma 116 (note the hint in the lecture notes): Let \( \{e_j\} = \{e_1, ..., e_n\} \) be a basis for \( \mathbb{R} \), and let \( \{b_j\} = \{b_1, ..., b_m\} \) be any set of vectors belonging to \( \mathbb{R} \) with \( m > n \). Then \( \{b_j\} \) can not be linearly independent.

5. Lecture Notes Exercise 124: Using the 'fundamental theorem of algebra' and the fact that \( \text{rank}(A) = \text{rank}(A') \), show that

\[
\text{rank}(A) + \text{null}(A') = m \\
\text{null}(A) - \text{null}(A') = n - m
\]

6. Lecture Notes Exercise 129: Using the properties of transpose and inverse:

(a) Prove that \( A^{-k} = (A^k)^{-1} \)

(b) Consider the matrix \( Z = X(X'X)^{-1}X' \) where \( X \) is an arbitrary \( m \times n \) matrix. Under what conditions on \( X \) is \( Z \) well-defined? Show that \( Z \) is symmetric. Also show that \( ZZ = Z \) (i.e., that \( Z \) is idempotent).

(Note: this is another repeat, but this one I included simply because it really is worth doing - you will use these facts, and the techniques needed to prove them, a LOT in statistics and econometrics, so it would be helpful to get them down now.)

7. Lecture Notes Exercise 150: Show that, if \( [A, b] \) is singular, then and only then \( X^* = \emptyset \), and further \( \dim(X^*) = \text{null}([A, b]) - 1 \).

8. Calculate \( e^A \) for \( A \) equal to

(a) \( \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix} \) (hint: diagonalize!)

(b) \( \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix} \) (hint: start with \( A^2 \), and recall that \( e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \))

9. Lecture Notes Exercise 182: Let \( X \) be an \( m \times n \) matrix with \( m \geq n \) and \( \text{rk}(X) = n \). Show that \( X'X \) is positive definite.

10. Lecture Notes Exercise 183: Show that a positive definite matrix is non-singular.

(Conclude from the past two exercises that so long as \( m \geq n \) and \( \text{rk}(X) = n \) - as you will generally assume when you estimate systems of equations - that you don’t need to wonder whether the term \( (X'X)^{-1} \) is defined.)