Lectures 10 -11
Repeated Games

14.12 Game Theory
Muhamet Yildiz

Road Map
1. Forward Induction – Examples
2. Finitely Repeated Games with observable actions
   1. Entry-Deterrence/Chain-store paradox
   2. Repeated Prisoners’ Dilemma
   3. A general result
   4. When there are multiple equilibria
3. Infinitely repeated games with observable actions
   1. Discounting / Present value
   2. Examples
   3. The Folk Theorem
   4. Repeated Prisoners’ Dilemma, revisited –tit for tat
   5. Repeated Cournot oligopoly
Forward Induction

**Strong belief in rationality:** At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies $s$ and $s'$ of a player $i$ that are consistent with a history of play, and if $s$ is strictly dominated but $s'$ is not, at this history no player $j$ believes that $i$ plays $s$.)

---

Burning Money

```
B S
---
B 3,1 .1,.1
S .1,.1 1,3

B S
---
B 2,1 -.9,.1
S -.9,.1 0,3

BB BS SB SS
---
0B 3,1 3,1 .1,.1 .1,.1
0S .1,.1 .1,.1 1,3 1,3
DB 2,1 -.9,.1 2,1 -.9,.1
DS -.9,.1 0,3 -.9,.1 0,3
O T H E R
```
Repeated Games

Entry deterrence

1 Enter 2 Acc. (1,1)

X
(0,2) Fight (-1,-1)
Entry deterrence, repeated twice, many times

What would happen if repeated n times?

Prisoners’ Dilemma, repeated twice, many times

- Two dates $T = \{0,1\};$
- At each date the prisoners’ dilemma is played:

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>5,5</td>
<td>0,6</td>
</tr>
<tr>
<td>D</td>
<td>6,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- At the beginning of 1 players observe the strategies at 0. Payoffs= sum of stage payoffs.
What would happen if $T = \{0,1,2,\ldots,n\}$?

### A general result

- $G$ = “stage game” = a finite game
- $T = \{0,1,\ldots,n\}$
- At each $t$ in $T$, $G$ is played, and players remember which actions taken before $t$;
- Payoffs = Sum of payoffs in the stage game.
- Call this game $G(T)$.

**Theorem:** If $G$ has a unique subgame-perfect equilibrium $s^*$, $G(T)$ has a unique subgame-perfect equilibrium, in which $s^*$ is played at each stage.
With multiple equilibria

\[ T = \{0,1\} \]

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>M2</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>1,1</td>
<td>5,0</td>
<td>0,0</td>
</tr>
<tr>
<td>M1</td>
<td>0,5</td>
<td>4,4</td>
<td>0,0</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>0,0</td>
<td>3,3</td>
</tr>
</tbody>
</table>

Infinitely repeated Games with observable actions

- \( T = \{0,1,2,\ldots,t,\ldots\} \)
- \( G = \) “stage game” = a finite game
- At each \( t \) in \( T \), \( G \) is played, and players remember which actions taken before \( t \);
- Payoffs = Discounted sum of payoffs in the stage game.
- Call this game \( G(T) \).
Definitions

The *Present Value* of a given payoff stream $\pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots)$ is

$$PV(\pi; \delta) = \sum_{t=1}^{\infty} \delta^t \pi_t = \pi_0 + \delta \pi_1 + \ldots + \delta^t \pi_t + \ldots$$

The *Average Value* of a given payoff stream $\pi$ is

$$(1-\delta)PV(\pi; \delta) = (1-\delta)\sum_{t=1}^{\infty} \delta^t \pi_t$$

The *Present Value* of a given payoff stream $\pi$ at $t$ is

$$PV_t(\pi; \delta) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_s = \pi_t + \delta \pi_{t+1} + \ldots + \delta^s \pi_{t+s} + \ldots$$

Infinite-period entry deterrence

1. **Enter**
2. **Acc.**

Strategy of Entrant:
- Enter iff
- Accomodated before.

Strategy of Incumbent:
- Accommodate iff
- Accomodated before.
Incumbent:
• $V(\text{Acc.}) = V_A =$
• $V(\text{Fight}) = V_F =$
• Case 1: Accommodated before.
  – Fight =>
  – Acc. =>
• Case 2: Not Accommodated
  – Fight =>
  – Acc. =>
  – Fight ⇔

Entrant:
• Accommodated
  – Enter =>
  – X =>
• Not Acc.
  – Enter =>
  – X =>

Infinitely-repeated PD

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>5,5</td>
<td>0,6</td>
</tr>
<tr>
<td>D</td>
<td>6,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

A Grimm Strategy:
Defect iff someone defected before.

• $V_D = 1/(1-\delta)$;
• $V_C = 5/(1-\delta) = 5V_D$;
• Defected before (easy)
• Not defected
  – D =>
  – C =>
  – C ⇔
Tit for Tat

- Start with C; thereafter, play what the other player played in the previous round.
- Is (Tit-for-tat,Tit-for-tat) a SPE?

- Modified: Start with C; if any player plays D when the previous play is (C,C), play D in the next period, then switch back to C.

Folk Theorem

**Definition:** A payoff vector \( v = (v_1, v_2, \ldots, v_n) \) is feasible iff \( v \) is a convex combination of some pure-strategy payoff-vectors, i.e.,
\[
v = p_1u(a^1) + p_2u(a^2) + \ldots + p_ku(a^k),
\]
where \( p_1 + p_2 + \ldots + p_k = 1 \), and \( u(a^j) \) is the payoff vector at strategy profile \( a^j \) of the stage game.

**Theorem:** Let \( x = (x_1, x_2, \ldots, x_n) \) be a feasible payoff vector, and \( e = (e_1, e_2, \ldots, e_n) \) be a payoff vector at some equilibrium of the stage game such that \( x_i > e_i \) for each \( i \). Then, there exist \( \delta < 1 \) and a strategy profile \( s \) such that \( s \) yields \( x \) as the expected average-payoff vector and is a SPE whenever \( \delta > \delta \).
Folk Theorem in PD

<table>
<thead>
<tr>
<th></th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>5,5</td>
<td>0,6</td>
</tr>
<tr>
<td>D</td>
<td>6,0</td>
<td>1,1</td>
</tr>
</tbody>
</table>

- A SPE with PV (1.1,1.1)?
  - With PV (1.1,5)?
  - With PV (6,0)?
  - With PV (5.9,0.1)?

Infinitely-repeated Cournot oligopoly

- N firms, MC = 0; P = max{1-Q,0};
- Strategy: Each is to produce \( q = 1/(2n) \); if any firm defects produce \( q = 1/(1+n) \) forever.
- \( V_C = \)
- \( V_D = \)
- \( V(D|C) = \)
- Equilibrium \( \Leftrightarrow \)
IRCD (n=2)

- Strategy: Each firm is to produce $q^*$; if any one deviates, each produce $1/(n+1)$ thereafter.
- $V_C = q^*(1-2q^*)/(1-\delta)$;
- $V_D = 1/(9(1-\delta))$;
- $V_{D/C} = \max q(1-q^*)-q + \delta V_D = (1-q^*)^2 / 4 + \frac{\delta}{9(1-\delta)}$
- Equilibrium iff
  \[
  q^* (1-2q^*) \geq (1-\delta)(1-q^*)^2 / 4 + \delta / 9
  \]
- $\Leftrightarrow q^* \geq \frac{9 - 5\delta}{3(9 - \delta)}$

Graph:
- $x = \delta$, $y = (3 - 3\delta)/(9 - \delta)$
- $0 \leq x \leq 1$, $0 \leq y \leq 0.4$
Carrot and Stick

Produce $\frac{1}{4}$ at the beginning; at ant $t > 0$, produce $\frac{1}{4}$ if both produced $\frac{1}{4}$ or both produced $x$ at $t-1$; otherwise, produce $x$.

Two Phase: Cartel & Punishment

$V_C = \frac{1}{8}(1-\delta)$. $V_x = x(1-2x) + \delta V_C$.

$V_{DC} = \max q(1-1/4-q) + \delta V_x = (3/8)^2 + \delta V_x$

$V_{DX} = \max q(1-x-q) + \delta V_x = (1-x)^2/4 + \delta V_x$

$V_C \geq V_{DC} \iff V_C \geq (3/8)^2 + \delta^2 V_C + \delta x(1-2x)$

$\Leftrightarrow (1-\delta^2) V_C - (3/8)^2 \geq \delta x(1-2x) \iff (1+\delta)/8 - (3/8)^2 \geq \delta x(1-2x)$

$V_X \geq V_{DC} \iff (1-\delta)V_X \geq (1-x)^2/4 \iff (1-\delta)(x(1-2x) + \delta/8(1-\delta)) \geq (1-x)^2/4$

$\Leftrightarrow (1-\delta)x(1-2x) + \delta/8 \geq (1-x)^2/4$

$2x^2 - x + 1/8 - 9/64\delta \geq 0$

$(9/4-2\delta)x^2 - (3-2\delta)x + \delta/8(1-\delta) \leq 0$