Lectures 10 -11
Repeated Games

14.12 Game Theory
Muhamet Yildiz

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2. Finitely Repeated Games with observable actions
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Forward Induction

**Strong belief in rationality:** At any history of the game, each agent is assumed to be rational if possible. (That is, if there are two strategies $s$ and $s'$ of a player $i$ that are consistent with a history of play, and if $s$ is strictly dominated but $s'$ is not, at this history no player $j$ believes that $i$ plays $s$.)

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**Burning Money**

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Repeated Games

Entry deterrence

1 Enter
X
(0,2)

2 Acc.

Fight
(-1,-1)

(1,1)
Entry deterrence, repeated twice, many times

What would happen if repeated n times?

Prisoners’ Dilemma, repeated twice, many times

- Two dates $T = \{0, 1\}$;
- At each date the prisoners’ dilemma is played:

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<tr>
<td>D</td>
<td>6,0</td>
<td>1,1</td>
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- At the beginning of 1 players observe the strategies at 0. Payoffs = sum of stage payoffs.
A general result

- $G$ = “stage game” = a finite game
- $T = \{0, 1, \ldots, n\}$
- At each $t$ in $T$, $G$ is played, and players remember which actions taken before $t$;
- Payoffs = Sum of payoffs in the stage game.
- Call this game $G(T)$.

**Theorem:** If $G$ has a unique subgame-perfect equilibrium $s^*$, $G(T)$ has a unique subgame-perfect equilibrium, in which $s^*$ is played at each stage.
With multiple equilibria

\[ T = \{0,1\} \]

\[
\begin{array}{c|ccc}
  & L & M2 & R \\
\hline
T & 1,1 & 5,0 & 0,0 \\
M1 & 0,5 & 4,4 & 0,0 \\
B  & 0,0 & 0,0 & 3,3 \\
\end{array}
\]

\[ s^* = \]
- At \( t = 0 \), each \( i \) play \( M_i \);
- At \( t = 1 \), play \( (B,R) \) if \( (M_1,M_2) \)
at \( t = 0 \), play \( (T,L) \) otherwise.

\[
\begin{array}{c|ccc}
  & L & M2 & R \\
\hline
T & 2,2 & 6,1 & 1,1 \\
M1 & 1,6 & 7,7 & 1,1 \\
B  & 1,1 & 1,1 & 4,4 \\
\end{array}
\]

Infinitely repeated Games with observable actions

- \( T = \{0,1,2,\ldots,t,\ldots\} \)
- \( G = \) “stage game” = a finite game
- At each \( t \) in \( T \), \( G \) is played, and players remember which actions taken before \( t \);
- Payoffs = Discounted sum of payoffs in the stage game.
- Call this game \( G(T) \).
Definitions

The Present Value of a given payoff stream \( \pi = (\pi_0, \pi_1, \ldots, \pi_t, \ldots) \) is

\[
PV(\pi; \delta) = \sum_{t=1}^{\infty} \delta^t \pi_t = \pi_0 + \delta \pi_1 + \ldots + \delta^t \pi_t + \ldots
\]

The Average Value of a given payoff stream \( \pi \) is

\[
(1-\delta)PV(\pi; \delta) = (1-\delta)\sum_{t=1}^{\infty} \delta^t \pi_t
\]

The Present Value of a given payoff stream \( \pi \) at \( t \) is

\[
PV_t(\pi; \delta) = \sum_{s=t}^{\infty} \delta^{s-t} \pi_s = \pi_t + \delta \pi_{t+1} + \ldots + \delta^s \pi_{t+s} + \ldots
\]

Infinite-period entry deterrence

\begin{center}
\begin{tabular}{c c c}
& Enter & 2 Acc. \\
1 & & (1,1) \\
X & |& |
\end{tabular}
\end{center}

Strategy of Entrant:
Enter iff
Accomodated before.

Strategy of Incumbent:
Accomodate iff
accomodated before.
Incumbent:
• \( V(\text{Acc.}) = V_A = 1/(1-\delta); \)
• \( V(\text{Fight}) = V_F = 2/(1-\delta); \)
• Case 1: Accommodated before.
  – Fight => -1 + \( \delta V_A \)
  – Acc. => 1 + \( \delta V_A \).
• Case 2: Not Accommodated
  – Fight => -1 + \( \delta V_F \)
  – Acc. => 1 + \( \delta V_A \)
  – Fight \( \Leftrightarrow -1 + \delta V_F \geq 1 + \delta V_A \)
  \( \Leftrightarrow V_F - V_A = 1/(1-\delta) \geq 2/\delta \)
  \( \Leftrightarrow \delta \geq 2/3. \)

Entrant:
• Accommodated
  – Enter => 1+\( V_{AE} \)
  – X => 0 +\( V_{AE} \)
• Not Acc.
  – Enter =>-1+\( V_{FE} \)
  – X => 0 +\( V_{FE} \)

Infinitely-repeated PD

\[
\begin{array}{c|cc}
  & C & D \\
\hline
  C & 5,5 & 0,6 \\
  D & 6,0 & 1,1 \\
\end{array}
\]

• \( V_D = 1/(1-\delta); \)
• \( V_C = 5/(1-\delta) = 5V_D; \)
• Defected before (easy)
• Not defected

A Grimm Strategy:
Defect iff someone defected before.

– D =>
– C =>
– C \( \Leftrightarrow \)
Tit for Tat

- Start with C; thereafter, play what the other player played in the previous round.
- Is (Tit-for-tat,Tit-for-tat) a SPE?

- **Modified:** There are two modes:
  1. Cooperation, when play C, and
  2. Punishment, when play D.

  Start in Cooperation; if any player plays D in Cooperation mode, then switch to Punishment mode for one period and switch back to the Cooperation period next.

Folk Theorem

**Definition:** A payoff vector \( v = (v_1,v_2,...,v_n) \) is feasible iff \( v \) is a convex combination of some pure-strategy payoff-vectors, i.e.,
\[
v = p_1u(a^1) + p_2u(a^2) + ... + p_ku(a^k),
\]
where \( p_1 + p_2 + ... + p_k = 1 \), and \( u(a^j) \) is the payoff vector at strategy profile \( a^j \) of the stage game.

**Theorem:** Let \( x = (x_1,x_2,...,x_n) \) be a feasible payoff vector, and \( e = (e_1,e_2,...,e_n) \) be a payoff vector at some equilibrium of the stage game such that \( x_i > e_i \) for each \( i \). Then, there exist \( \delta < 1 \) and a strategy profile \( s \) such that \( s \) yields \( x \) as the expected average-payoff vector and is a SPE whenever \( \delta > \tilde{\delta} \).
Folk Theorem in PD

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• A SPE with PV (1.1,1.1)?
  – With PV (1.1,5)?
  – With PV (6,0)?
  – With PV (5.9,0.1)?

Infinitely-repeated Cournot oligopoly

• N firms, MC = 0; P = max{1-Q,0};
• Strategy: Each is to produce \( q = 1/(2n) \); if any firm defects produce \( q = 1/(1+n) \) forever.

• \( V_C = \)
• \( V_D = \)
• \( V(D|C) = \)
• Equilibrium \( \Leftrightarrow \)
IRCD (n=2)

- Strategy: Each firm is to produce $q^*$; if any one deviates, each produce $1/(n+1)$ thereafter.
- $V_C = q^*(1-2q^*)/(1-\delta)$;
- $V_D = 1/(9(1-\delta))$;
- $V_{D/C} = \max q(1-q^*-q) + \delta V_D = (1-q^*)^2 / 4 + \frac{\delta}{9(1-\delta)}$
- Equilibrium iff

$$q^*(1-2q^*) \geq (1-\delta)(1-q^*)^2 / 4 + \delta / 9$$

- $q^* \geq \frac{9 - 5\delta}{3(9 - \delta)}$

$x=\delta, y=(3-3\delta)(9-\delta)$
Carrot and Stick

Produce $\frac{1}{4}$ at the beginning; at ant $t > 0$, produce $\frac{1}{4}$ if both produced $\frac{1}{4}$ or both produced $x$ at $t-1$; otherwise, produce $x$.

Two Phase: Cartel & Punishment

$V_C = \frac{1}{8}(1-\delta)$. $V_x = x(1-2x) + \delta V_C$.

$V_{DC} = \max q(1-1/4-q) + \delta V_x = (3/8)^2 + \delta V_x$

$V_{DX} = \max q(1-x-q) + \delta V_x = (1-x)^2/4 + \delta V_x$

$V_C \geq V_{DC} \iff V_C \geq (3/8)^2 + \delta^2 V_C + \delta x(1-2x)$

$\Leftrightarrow (1-\delta^2) V_C - (3/8)^2 \geq \delta x(1-2x) \Leftrightarrow (1+\delta)/8 - (3/8)^2 \geq \delta x(1-2x)$

$V_X \geq V_{DX} \Leftrightarrow (1-\delta) V_x \geq (1-x)^2/4 \Leftrightarrow (1-\delta)(x(1-2x) + \delta/8(1-\delta)) \geq (1-x)^2/4$

$\Leftrightarrow (1-\delta)x(1-2x) + \delta/8 \geq (1-x)^2/4$

\[2x^2 - x + 1/8 - 9/64x \geq 0\]

\[(9/4-2\delta)x^2 - (3-2\delta)x + \delta/8(1-\delta) \leq 0\]

Incomplete information
Incomplete information

We have incomplete (or asymmetric) information if one player knows something (relevant) that some other player does not know.

### An Example

<table>
<thead>
<tr>
<th>Nature</th>
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<th>Low 1-p</th>
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<tbody>
<tr>
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<td></td>
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<tr>
<td>W</td>
<td>Work</td>
<td>(1, 2)</td>
</tr>
<tr>
<td>Shirk</td>
<td>Shirk</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>Hire</td>
<td>Hire</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Do not hire</td>
<td>-</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>W</td>
<td>Work</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>Shirk</td>
<td>Shirk</td>
<td>(-1, 2)</td>
</tr>
<tr>
<td>Hire</td>
<td>Hire</td>
<td>(0, 0)</td>
</tr>
<tr>
<td>Do not hire</td>
<td>-</td>
<td>(0, 0)</td>
</tr>
</tbody>
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The same example

Another Example

What would you ask if you were to choose p from [0,4]?
Same “Another Example”

Bayes’ Rule

\[
\text{Prob}(A \text{ and } B) = \frac{\text{Prob}(A|B)\text{Prob}(B)}{\text{Prob}(B)} = \frac{\text{Prob}(B|A)\text{Prob}(A)}{\text{Prob}(B)}
\]
Example

- \( \text{Prob(Work|Success)} = \frac{\mu p}{\mu p + (1-\mu)(1-p)} \)
- \( \text{Prob(Work|Failure)} = \frac{(1-\mu)p}{\mu(1-p) + (1-\mu)p} \)

Diagram:

- Work
- Success
- Failure
- Shirk
- \( \mu \)
- \( 1-\mu \)
- \( p \)
- \( 1-p \)

Graph:

- \( P(W|S) \)
- \( P(W|F) \)