1. Nash Implementation

Consider a public good problem. There are \( I \geq 3 \) agents. \( x \) is 1 if the good is supplied and 0 if not and \( t^i \in \mathcal{R} \) is the transfer to agent \( i \). The preferences of agent \( i \) are quasi-linear: \( \theta^i x + t^i \) with \( \theta^i \in \mathcal{R} \). We want to study social choice correspondences, \( f \), that we can implement in Nash equilibrium.

(a). Show that monotonicity implies that \( f \) satisfies the following two conditions.

**Condition 1:** Consider \( \theta \) and \( (x, t^1, ..., t^I) \in f(\theta) \) such that \( x = 1 \). Consider also \( \phi \) such that \( \phi \geq \theta \) (this means that \( \forall i, \phi^i \geq \theta^i \)). We then have \( (x, t^1, ..., t^I) \in f(\phi) \).

**Condition 2:** Consider \( \theta \) and \( (x, t^1, ..., t^I) \in f(\theta) \) such that \( x = 0 \). Consider also \( \phi \) such that \( \theta \geq \phi \) (this means that \( \forall i, \theta^i \geq \phi^i \)). We then have \( (x, t^1, ..., t^I) \in f(\phi) \).

(b). Consider now \( f \) that satisfies conditions 1 and 2. Show that it is monotonic.

(c). Show that \( f \) satisfies no veto power. Conclude that \( f \) is implementable in Nash equilibrium.

(d). Show that we can implement an \( f \) that satisfies efficient supply (i.e. \( \forall \theta \) and \( \forall (x, t^1, ..., t^I) \in f(\theta) \), \( x = 1 \) if and only if \( \sum_{i=1}^I \theta^i \geq 0 \)), that does not involve transfers when \( x = 0 \) and that is balanced (i.e. \( \forall \theta \) and \( \forall (x, t^1, ..., t^I) \in f(\theta) \), \( \sum_{i=1}^I t^i = 0 \)).

(e). Argue that our results are very satisfactory if we only care about efficiency (efficient supply and not throwing money away) but much less satisfactory if we care about a “fair” sharing of the cost of the public good.

2. Dominant Strategy versus Nash Implementation

This problem gives an example of a social choice function that is not implementable in dominant strategies but is in Nash equilibrium.

Suppose that there are 3 agents, 1, 2 and 3. There are five outcomes, \( a, b, c, d \) and \( e \). Agent 1’s preferences are indexed by \( \theta^1 \). \( \theta^1 \) can take two values, \( \theta_1^1 \) or \( \theta_2^1 \). If \( \theta^1 = \theta_1^1 \) preferences are \( a \succ b \succ c \succ d \succ e \). If \( \theta^1 = \theta_2^1 \) preferences are instead \( b \succ a \succ e \succ d \succ c \). Agent 2’s preferences are indexed by \( \theta^2 \). \( \theta^2 \) can also take two values, \( \theta_1^2 \) or \( \theta_2^2 \). If \( \theta^1 = \theta_1^2 \) preferences are \( a \succ b \succ c \succ d \succ e \). If \( \theta^1 = \theta_2^2 \) preferences are instead \( b \succ a \succ e \succ d \succ c \). Finally, there is no uncertainty about the preferences of agent 3. These are \( e \succ d \succ c \succ b \succ a \). All agents know each other’s preferences.
We define a social choice function \( f \) as follows

\[
\begin{align*}
 f(\theta_1^1, \theta_2^1) &= a, \\
 f(\theta_1^1, \theta_2^2) &= d, \\
 f(\theta_1^2, \theta_2^1) &= d, \\
 f(\theta_1^2, \theta_2^2) &= b.
\end{align*}
\]

(a) Show that \( f \) cannot be fully implemented in dominant strategies.
(b) Show that \( f \) is monotonic.
(c) Show that \( f \) satisfies no veto power.
(d) Conclude that \( f \) is implementable in Nash equilibrium.

3. Risk sharing and implementation with renegotiation-design
Consider an implementation setup without investment but with risk-sharing. Assume the parties, a buyer and a seller, have the following utility functions:

\[
\begin{align*}
 u_b(v(q, \theta) - p) \\
 u_s(p - k(q, \theta))
\end{align*}
\]

where \( u_b, v \) and \( u_s \) are increasing and concave functions, \( k \) is an increasing and convex function, and \( v(q, 0) = 0 \) and \( k(q, 0) = 0 \) for all \( \theta \)'s. Contracting takes place before \( \theta \) is known, trade \((p, q)\) after \( \theta \) has been observed by both parties.

(a) describe the first best in this setup;
(b) assume that \( \theta \) is observable but unverifiable, and that the parties cannot commit not to renegotiate but can contractually agree on message-contingent default options and allocations of the entire bargaining power to one party. Construct a revelation game which implements the first best without equilibrium renegotiation.
(c) can a contract without messages but with equilibrium renegotiation implement the first best ? Interpret.

4. ‘Incomplete contracts’
Consider a model of a seller who can invest in quality enhancements, \( r \), and produce \( q \) units at a price of \( cq \). The seller’s utility function is

\[
U^S = t - cq - r,
\]

where \( t \) is the net transfer to the seller from the buyer. The buyer values production and quality according to the relationship

\[
U^B = u(q, r) - t,
\]

where \( u_{qr} > 0 \). Throughout, quantity and transfers are assumed contractible, quality is not. The timing is as follows. At date 0, the parties can write a contract to be specified below. At date 1, the seller chooses observable but
unverifiable quality, $r$. At date 2, the buyer can make a credible take-it-or-leave-it renegotiation offer to the seller for a quantity $q$ and transfer $t$. At date 3, the seller can accept the buyer’s new date 2 offer, ignore the date 2 offer and supply at the terms of the original date 0 contract if they yield a higher payoff, or walk away. Thus, think of the date 0 contract as legally binding, but renegotiable if both parties agree.

(a) What is the first-best quantity and quality?

(b) Suppose that no contract can be written at date 0 (i.e., the default contract is implicitly $(q, t) = (0, 0)$). What is the equilibrium level of quality? Are $q$ and $r$ set inefficiently or efficiently (i.e., is the first-order condition in (a) satisfied for the variable)?

(c) Suppose that a contract can be written at date 0 which is a single positive quantity-transfer pair, $(q, t)$, at which trade must take place if the seller desires at date 3. What is the equilibrium level of quality? Are $q$ and $r$ set inefficiently or efficiently?

(d) Suppose the contract at date gives the buyer more freedom than the one in (c). Specifically, suppose that the contract is an offering of two choices to the buyer: a positive quantity-transfer pair, $(q, t)$, and an option to walk away, $(0, 0)$. As before, the buyer must accept one of these conditions at date 3 if the seller forces him to honor the date 0 contract. Importantly, the buyer now has the option to walk away built into the date 0 contract. What is the equilibrium level of quality? Are $q$ and $r$ set inefficiently or efficiently?

Remark on this exercise: Bernheim and Whinston have argued that the result in (d) is an example where an “incomplete” can outperform a “complete” contract. Note that their definition of a complete contract is somewhat atypical — a complete contract specifies exactly the equilibrium levels of all contractible variables with no additional freedom or exclusion. Hence, no out-of-equilibrium play is allowed. But we know that off-equilibrium-path play is sometimes crucial to hold together equilibria, so it is not surprising that “completeness” in the Bernheim-Whinston sense may have a cost.

(e) Suppose instead that the model is one of cost-reduction by the seller rather than quality improvement. Specifically, let

$$U^B = u(q) - t$$

and

$$U^S = t - c(r)q - r,$$

where $c'(r) < 0$. Show that now a “complete” contract (i.e., in the Bernheim-Whinston sense: $(q, t)$) is optimal.

5. Advocates

Consider a principal who has to take a decision $d \in \{A, 0, B\}$. The optimal decision depends on a random parameter $\theta = \theta_A + \theta_B$. Ex ante, we have $\Pr(\theta_A =$
$-1) = \alpha$ and $\Pr(\theta_A = 0) = 1 - \alpha$ while $\Pr(\theta_B = 1) = \alpha$ and $\Pr(\theta_B = 0) = 1 - \alpha$. Moreover, $\theta_A$ and $\theta_B$ are independently distributed. Assume the optimal decision for the organization is $A$ if $\theta = -1$, $0$ if $\theta = 0$ and $B$ if $\theta = 1$. The losses from taking the wrong decisions are:

- $L_I > 0$, a "loss from inertia", if decision 0 is taken while $\theta \in \{-1, 1\}$.
- $L_E > 0$, a "loss from extremism", if decision $A$ or $B$ are taken while $\theta = 0$.
- $L_M > 0$, a "loss from misguided activism", if decision $A$ (respectively $B$) is taken while $\theta = 1$ (respectively if $\theta = -1$).

The principal has access to a population of risk-neutral agents who, by spending one unit of effort looking at $\theta_i$ ($i = A, B$), can obtain hard evidence that $|\theta_i| = 1$. Specifically, if $\theta_i = 0$, no evidence is ever found, while if $|\theta_i| = 1$, hard evidence of this is found with probability $q$. Agents’ utility functions are $w - nK$, where $w$ is the wage, $n$ is the number of units of effort expended by the agent ($n = 2$ means that the agent looks at both $\theta_A$ and $\theta_B$) and $K$ the cost of one unit of effort.

Assume that losses are such that: (i) if only one piece of evidence has been found but effort has been expended in both directions, it is optimal for the principal to move away from the zero decision; (ii) if no piece of evidence has been found but effort has been expended in both directions, it is optimal for the principal not to move away from the zero decision.

(a) what is the first best in this case? When is it optimal to have two units of effort expended?

(b) assume agents privately choose effort but can be paid per piece of evidence provided. Can the first best be achieved? Does it matter whether one or two agents (expending respectively two or one unit of effort) are hired by the principal? Does this depend on agents’ limited liability constraints?

(c) assume now that the hard evidence is automatically transmitted to the principal but that contracts can only be contingent on the decision taken by the principal, and not on the amount of information generated by the agent(s). What are the optimal contracts when a single agent is hired by the principal and when two agents are hired by the principal? Does this depend on agents’ limited liability constraints?

(d) assume finally that, whenever the hard evidence is generated, it is first obtained by the agent, who can then decide whether or not to communicate it to the principal (note however that it cannot be forged: but the agent can claim not to have learned anything). Does this change the conclusion reached in (c) above?