1 Groves and AGV Mechanisms (5 pts.)

We are limiting our attention to direct revelation mechanisms. A revelation mechanism is denoted by \( \{ \Theta, K(m), \{ z_i(m) \}_{i=1}^n \} \), where \( \Theta = \Theta_1 \times \cdots \times \Theta_n \) and \( \Theta_i \) is the message space of agent \( i \) (This is equivalent to the type space because we are considering direct mechanisms), \( K(m) \) is the decision and \( z_i(m) \) is the payment from agent \( i \) to the principal when \( m = (m_1, \ldots, m_n) \) is announced.

(a) A direct mechanism yielding truth-telling as a dominant strategy means

\[
\theta_i \in \arg \max_{m_i} v_i(K(m_i, m_{-i}); \theta_i) - z_i(m_i, m_{-i}) \quad \forall \theta_i, m_{-i}, i.
\]

This is equivalent to

\[
v_i(K(\theta_i, m_{-i}); \theta_i) - z_i(\theta_i, m_{-i}) \geq v_i(K(m_i, m_{-i}); \theta_i) - z_i(m_i, m_{-i}) \quad \forall m_i, m_{-i}, \theta_i, i.
\]

This implies

\[
v_i(K(\theta_i, \theta_{-i}); \theta_i) - z_i(\theta_i, \theta_{-i}) \geq v_i(K(m_i, \theta_{-i}); \theta_i) - z_i(m_i, \theta_{-i}) \quad \forall m_i, \theta_{-i}, \theta_i, i.
\]

If we assume all other agents are announcing the true types, by taking expectations over \( \theta_i \),

\[
E_{\theta_{-i}}[v_i(K(\theta_i, \theta_{-i}); \theta_i) - z_i(\theta_i, \theta_{-i})] \geq E_{\theta_{-i}}[v_i(K(m_i, \theta_{-i}); \theta_i) - z_i(m_i, \theta_{-i})] \quad \forall m_i, \theta_i, i,
\]

i.e.,

\[
\theta_i \in \arg \max_{m_i} E_{\theta_{-i}}[v_i(K(m_i, \theta_{-i}); \theta_i) - z_i(m_i, \theta_{-i})] \quad \forall \theta_i, i.
\]

This means truth telling constitutes Bayesian equilibrium.

(b) From the result in part (a), Groves mechanism of the form

\[
t_i(m) = \sum_{j \neq i} v_j(K^*(m) : m_j) + h_i(m_{-i})
\]
yields truth telling as a Bayesian equilibrium. This means

$$\theta_i \in \arg\max_{m_i} E_{\theta-i}[v_i(K(m_i, \theta_{-i}); \theta_i)] + E_{\theta-i}[\sum_{j \neq i} v_j(K^*(m_i, \theta_{-i}); \theta_j)].$$

Let

$$\psi_i(m_i) = E_{\theta-i}[\sum_{j \neq i} v_j(K^*(m_i, \theta_{-i}); \theta_j)].$$

Now consider a transfer scheme of the form

$$\tilde{t}_i(m) = \psi_i(m_i) + \tilde{h}_i(m_{-i}).$$

Obviously, this also yields truth telling as a Bayesian equilibrium. Notice that

$$\psi_i(m_i)$$

depends only on agent i’s own announcement and not others’. We want to choose $$\tilde{h}_i(m_i)$$ so that budget is balanced. We can do this by

$$\tilde{h}_i(m_{-i}) = \frac{-1}{n-1} \sum_{j \neq i} \psi_j(m_j).$$

To check that budget is actually balanced, observe

$$\sum_i \tilde{t}_i(m) = \sum_i \psi_i(m_i) + \sum_i \tilde{h}_i(m_{-i})$$

$$= \sum_i \psi_i(m_i) + \sum_i \sum_{j \neq i} \frac{-1}{n-1} \psi_j(m_j)$$

$$= \sum_i \psi_i(m_i) - \sum_i \psi_i(m_i) = 0.$$

This mechanism is called d’Aspremont Gerard-Varet mechanism.

2 No Trade Theorem (5 pts.)

The proof presented here is essentially Theorem 1 of Holmstrom and Myerson (1983, *Econometrica*), which is closest to the model of the problem.

To solve this problem, we need to be more precise about how or under what conditions recontracting takes place at $$t = 1$$. The equilibrium outcome could be sensitive to who has the bargaining power in the recontracting stage. However, to prove this result, it is sufficient to assume the following (relatively weak) assumption.

**Assumption** If agents agree to recontract on $$d'(\theta)$$ from the original contract $$d^0(\theta)$$, it is common knowledge that all agents weakly prefer $$d'(\theta)$$ to $$d(\theta)$$.
This is intuitive. If people agree to recontract, then an agent must know other agents are also happy to recontract, — this gives the agent some additional information about other people’s signal, and he must be still happy to recontract, given this additional information. And an agent must be happy to trade given he knows everyone else is happy to trade given every one else knows everyone is happy to trade..., and so on.

We also assume the following.

Assumption The original contract \( d^0(\theta) \) is *ex ante* Pareto efficient.

Assumption Agents are weakly risk averse.

Let us be more precise about what we mean by common knowledge at \( t = 1 \). Each agent’s information partition is essentially given by his signal. A common knowledge structure is given by common coarsening of agents’ information partition. We say an event \( R \) is common knowledge if \( R \) is an element of the common coarsening. Formally,

**Definition** An event \( R \) is common knowledge if \( R \) is of the form \( R = R_1 \times \cdots \times R_I \) where \( R_i \subset S_i \) for all \( i \), \( S_i \) is set of possible signals for agent \( i \) and
\[
p_i(\hat{s}_{-i}|s_i) = 0; \quad \forall s \in R, \forall \hat{s} \notin R, \forall i.
\]

This definition, together with the assumption on recontracting, tells us the following. If people recontract on \( d'(\theta) \) when signals are \( s = (s_1, \cdots, s_I) \), then there exists a common knowledge event \( R \) such that \( s \in R \) and
\[
U_i(d'(\theta)|s_i') \geq U_i(d^0(\theta)|s_i') \quad \forall s' \in R, \forall i.
\]

The claim can be formally stated as follows.

**Theorem** If agents agree to recontract on \( d'(\theta) \) from the original contract \( d^0(\theta) \), no agent strictly prefers \( d'(\theta) \) to \( d^0(\theta) \).

Now we are going to prove the theorem. In our problem, the decision rule can be contingent only on \( \theta \) and not on \( s \). However, it is useful to think what if we could have done if a decision rule could be contingent on \( s \). Let \( \delta(\theta, s) \) be a decision rule contingent on \( \theta \) and \( s \). Let \( \Delta \) be the set of such decision rules. Define \( \delta^0(\theta, s) = d^0(\theta) \forall s \forall \theta \) and \( \delta'(\theta, s) = d'(\theta) \forall s \forall \theta \). Observe agents’ payoffs depends only on \( d \) and \( \theta \) and not on \( s \). This means the efficient decision doesn’t depend on \( s \). Therefore, \( \delta^0(\theta, s) = d^0(\theta) \forall s \) is ex ante Pareto efficient even among \( \Delta \).

Consider a decision rule \( \hat{\delta}(\theta, s) \) defined by
\[
\hat{\delta}(\theta, s) = \begin{cases} 
\delta'(\theta, s) & s \in R \\
\delta^0(\theta, s) & s \notin R.
\end{cases}
\]
This decision rule gives higher expected payoff than $\delta^0(\theta, s)$ to each agent conditional on event $R$ and the same expected payoff, so ex ante expected payoffs are higher than $\delta^0(\theta, s)$ for all agents. This means $\delta(\theta, s)$ weakly Pareto dominates $\delta^0(\theta, s)$. Furthermore, if some agent strictly prefers trading conditional on event $R$, he gets strictly higher ex ante expected payoff from $\hat{\delta}(\theta, s)$, implying it strictly Pareto dominates $\delta^0(\theta, s)$. This contradicts $\delta^0(\theta, s) = d^0(\theta)\forall s$ being ex ante Pareto efficient.

Example: Two states $\theta \in \{\theta_1, \theta_2\}$, two decisions $d \in \{d_1, d_2\}$. Both agent gets utility of 100 if $\theta = \theta_1$ and $d = d_1$, or if $\theta = \theta_2$ and $d = d_2$. Otherwise, utility is 0 for both agents. $s_1 = s_2 = \theta$, that is, all agents learn the state without noise at $t = 1$. Ex ante probability of $\theta_1$ is 0.9 and probability of $\theta_2$ is 0.1. If state-contingent decision rule is infeasible, only feasible decision rules are to always choose $d_1$ and to always choose $d_2$. Ex ante optimal is to choose $d_1$. However, once $\theta_2$ happens and both agents observes it, they want to recontract on $d_2$.

3 Bilateral Trading (5 pts.)

Since $v_1 > c_1$ and $v_2 > c_2$, an efficient contract should always implement trade. Without loss of generality, assume $c_2 > c_1$. (You may think we cannot do that because we have assumption $\frac{1}{2}(v_1 + v_2) < c_2$, but in fact we only need to assume that $\frac{1}{2}(v_1 + v_2)$ is less than max of $c_1$ and $c_2$. This is implied by $\frac{1}{2}(v_1 + v_2) < c_2$ even if $c_1 > c_2$.)

Because of the revelation principle, we can limit our attention to direct mechanisms. A direct mechanism is $\{C, x, t\}$, where the buyer is asked to announce his type $\hat{c}$ from the message space $C \equiv \{c_1, c_2\}$, $x(\hat{c})$ is the probability of trade, and $t(\hat{c})$ is the transfer from the buyer to the seller. The IC constraints for the seller are

$$(1 - x(c_1))c_1 + t(c_1) \geq (1 - x(c_2))c_1 + t(c_2),$$
$$(1 - x(c_2))c_2 + t(c_2) \geq (1 - x(c_1))c_2 + t(c_1).$$

The IR constraints for the seller are

$$(1 - x(c_1))c_1 + t(c_1) \geq c_1,$$
$$(1 - x(c_2))c_2 + t(c_2) \geq c_2.$$

The IR constraints for the buyer is

$$\frac{1}{2}(x(c_1)v_1 - t(c_1)) + \frac{1}{2}(x(c_2)v_2 - t(c_2)) \geq 0.$$

Efficiency requires $x(c_1) = x(c_2) = 1$. Plugging this into constraints, we find $t(c_1) = t(c_2)$ from IC, and then $t(c_1) = t(c_2) \geq c_2$ from IR for the seller. The
IR for the buyer becomes \( \frac{1}{2}(v_1 + v_2) \geq \frac{1}{2}(t(c_1) + t(c_2)) \). Combining them,

\[
\frac{1}{2}(v_1 + v_2) \geq t(c_1) = t(c_2) \geq c_2.
\]

This is impossible under our assumption \( \frac{1}{2}(v_1 + v_2) < c_2 \).

The Myerson-Satterthwaite theorem says efficiency is not attained with budget balancing if (i) the seller and the buyer know their own valuation but not other’s, (ii) valuations are independently distributed, and (iii) both trading and not trading are efficient with positive probability.

In this problem, the information structure is different (buyer doesn’t know her valuation and valuations are perfectly correlated), and trading is always efficient. It is easy to show that efficiency is achievable if the buyer also learns her valuation.

4 Laffont-Tirole (8 pts.)

(a) The constraints are

\[
\begin{align*}
(\text{IC}) & \quad t(\theta) - \psi(x(\theta) - \theta) \geq t(\theta') - \psi(x(\theta') - \theta) \quad \forall \theta, \theta', \\
(\text{IR}) & \quad t(\theta) - \psi(x(\theta) - \theta) \geq 0 \quad \forall \theta.
\end{align*}
\]

Define \( U(\theta'|\theta) = t(\theta') - \psi(x(\theta') - \theta) \) and \( U(\theta) = U(\theta|\theta) \).

Characterization lemma The IC constraint is satisfied if and only if

(i) \( U(\theta_1) - U(\theta_0) = \int_{\theta_0}^{\theta_1} \psi'(x(s) - s)ds \) for all \( \theta_0, \theta_1 \).

(ii) \( x(\theta) \) is nondecreasing.

Proof of necessity Two IC constraints gives

\[
\psi(x(\theta) - \theta) - \psi(x(\theta') - \theta) \leq t(\theta) - t(\theta') \leq \psi(x(\theta) - \theta') - \psi(x(\theta') - \theta'),
\]

\[
\int_{x(\theta)}^{x(\theta')} \psi'(x - \theta)dx \leq \int_{x(\theta')}^{x(\theta)} \psi'(x - \theta')dx,
\]

\[
\int_{\theta}^{\theta'} \int_{x(\theta')}^{x(\theta)} \psi''(x - s)dxds \geq 0.
\]

Since \( \psi'' > 0, \theta > \theta' \) must imply \( x(\theta) \geq x(\theta') \). This proves \( x(\theta) \) is nondecreasing and thus it is continuous and differentiable almost everywhere. Now that we observed \( U(\theta'|\theta) \) is differentiable in \( \theta \), we can apply envelope theorem and get

\[
\frac{dU(\theta)}{d\theta} = U_2(\theta|\theta) = \psi'(x(\theta) - \theta).
\]

Note: The envelope theorem doesn’t require \( U \) to be differentiable in \( \theta' \). See Milgrom and Segal (Econometrica, 2002).
The fundamental theorem of calculus gives (i).

**Proof of sufficiency**

Suppose not. Then there exist \( \theta \) and \( \theta' \) such that

\[ U(\theta' \mid \theta) > U(\theta \mid \theta). \]

Just for simplicity, suppose \( \theta' < \theta \). We can make same discussion for the case where \( \theta' > \theta \). This implies

\[ U(\theta' \mid \theta) - U(\theta' \mid \theta') > U(\theta) - U(\theta'), \]

\[ \int_{\theta'}^{\theta} \psi'(x(\theta') - s)ds > \int_{\theta'}^{\theta} \psi'(x(s) - s)ds, \]

where the RHS comes from condition (i). Thus

\[ \int_{\theta'}^{\theta} \left( \psi'(x(\theta') - s)ds - \psi'(x(s) - s) \right)ds > 0. \]

However, \( \psi'(x(\theta') - s)ds - \psi'(x(s) - s) \) must be non-positive for all \( s \in [\theta', \theta] \) because \( x(\theta) \) is increasing and \( \psi > 0 \). Contradiction.

**Note:** We didn’t indeed assume differentiability of \( x \) or \( t \).

**(b)** Since we have transferable utility, the principal’s problem is to maximize

\[ E[s(x(\theta), \theta) - U(\theta)], \]

where, \( s(x(\theta), \theta) \) is the social surplus, i.e.,

\[ s(x(\theta), \theta) = x(\theta) - \psi(x(\theta) - \theta). \]

Now,

\[ E[U(\theta)] = \int_{\theta}^{\bar{\theta}} U(\theta)f(\theta)d\theta \]

\[ = -\int_{\theta}^{\bar{\theta}} U(\theta)(1 - F(\theta))'d\theta. \]

Integrating by parts,

\[ -U(\theta)(1 - F(\theta))\bigg|_{\theta}^{\bar{\theta}} + \int_{\theta}^{\bar{\theta}} \psi'(x(\theta) - \theta)(1 - F(\theta))d\theta. \]

The first part is 0 because \( U(\theta) = 0 \) and \( (1 - F(\theta)) = 0 \). The second part becomes

\[ \int_{\theta}^{\bar{\theta}} \psi'(x(\theta) - \theta) \frac{1 - F(\theta)}{f(\theta)} f(\theta)d\theta = E[\psi'(x(\theta) - \theta) \frac{1 - F(\theta)}{f(\theta)}]. \]
Now the maximization problem is reduced to
\[
\max_{x(\theta)} x(\theta) - \psi(x(\theta) - \theta) - \psi'(x(\theta) - \theta) \frac{1 - F(\theta)}{f(\theta)} \\
\text{s.t. } x(\theta) \text{ nondecreasing.}
\]

First, ignore the monotonicity constraint. Then the solution is given by maximizing pointwise. The FOC is
\[
1 - \psi'(x(\theta) - \theta) - \psi''(x(\theta) - \theta) \frac{1 - F(\theta)}{f(\theta)} = 0.
\]

Notice that the first best output level is given by \(\psi'(x(\theta) - \theta) = 1\) for all \(\theta\). The second best achieves efficiency only at the highest type and has output levels less than optimal for all other types.

Finally, we check the solution above satisfies monotonicity constraint. Let
\[
\pi = x - \psi(x - \theta) - \psi'(x - \theta) \frac{1 - F(\theta)}{f(\theta)}.
\]

A sufficient condition for \(x\) being nondecreasing is that \(\pi\) is supermodular in \(x\) and \(\theta\). The cross derivative is
\[
\frac{\partial^2 \pi}{\partial x \partial \theta} = \psi''(x - \theta) + \psi'''(x - \theta) \frac{1 - F(\theta)}{f(\theta)} - \psi''(x - \theta) \left(1 - \frac{1 - F(\theta)}{f(\theta)}\right) \leq 0.
\]

Thus \(x\) is in fact nondecreasing.

(c) Now we are considering Bayesian implementation, so each agent must, by truth-telling, be maximizing his expected payoff and be receiving at least outside utility level given other agents are truth-telling. Let \(U^i(\hat{\theta}_i|\theta_i)\) be agent \(i\)’s expected payoff from announcing \(\hat{\theta}_i\) when his true type is \(\theta_i\).

In this model,
\[
U^i(\hat{\theta}_i|\theta_i) = E_{\theta_{-i}}[t_i(\hat{\theta}_i, \theta_{-i}) - \phi_i(\hat{\theta}_i, \theta_{-i})\psi(x_i(\hat{\theta}_i, \theta_{-i}) - \theta_i)|\theta_i].
\]

The constraints are now
\[
(I\!C) \quad \theta_i \in \arg\max_{\theta_i} U^i(\hat{\theta}_i|\theta_i) \\
(IR) \quad U^i(\theta_i|\theta_i) \geq 0.
\]

If we assume differentiability of the mechanism, we can prove a generalized characterization lemma.

**Lemma** IC constraint is satisfied if and only if
\[
(i) \quad U^i(\theta_i^1) - U(\theta_i^0) = \int_{\theta_i^0}^{\theta_i^1} U^i_2(s|s)ds \text{ for all } \theta_i^0, \theta_i^1. \\
(ii) \quad U^i_2(\theta_i|\theta_i) \geq 0 \text{ for all } \theta_i.
\]
Proof Because of differentiability, IC constraint implies FOC and SOC, i.e., \( U_1^i(\theta_i|\theta_i) = 0 \) and \( U_{11}^i(\theta_i|\theta_i) \leq 0 \). Thus,
\[
\frac{dU^i(\theta_i|\theta_i)}{d\theta_i} = U_1^i(\theta_i|\theta_i) + U_2^i(\theta_i|\theta_i) = U_2^i(\theta_i|\theta_i),
\]
which is integrated up to obtain (i). If we differentiate the FOC above everywhere,
\[
\frac{dU^i(\theta_i|\theta_i)}{d\theta_i} = U_{11}^i(\theta_i|\theta_i) + U_{12}(\theta_i|\theta_i) = 0,
\]
and so the SOC implies \( U_{12}(\theta_i|\theta_i) \geq 0 \), which is (ii). This proves necessity.

Sufficiency can be proven in the same way as in part (a), and thus omitted.

By applying this lemma to our model, we get

Lemma IC constraint is satisfied if and only if
\[
U_i^i(\theta_i^0) - U_i^i(\theta_i) = \int_{\theta_i^0}^{\theta_i} E_{\theta_i}[\phi_i(\theta)\psi'(x(\theta) - \theta_i)])d\theta_i \forall \theta_i^0, \theta_i^1,
\]
\[
E_{\theta_i}[\phi_i(\theta)\psi''(x(\theta) - \theta_i)] \frac{\partial x_i}{\partial \theta_i} + \frac{\partial \phi_i}{\partial \theta_i} \psi'(x(\theta) - \theta_i) \geq 0 \forall \theta_i.
\]

(d) The objective function of the principal is now
\[
E_\theta \left[ \sum_{i=1}^{N} x_i(\theta) - \phi_i(\theta)\psi(x_i(\theta) - \theta_i) - U_i(\theta) \right].
\]

By integrating by parts and using \( U_i^i(\theta) = 0 \),
\[
E_{\theta_i}[U_i^i(\theta_i)] = E_{\theta_i} E_{\theta_i}[\phi_i(\theta)\psi'(x(\theta) - \theta_i)] \frac{1 - F(\theta_i)}{f(\theta_i)}.
\]

Substituting into the objective function and simplifying yields
\[
\max E \left[ \sum_{i=1}^{N} \phi_i(x_i - \psi(x_i - \theta_i) - \psi'(x_i - \theta_i) \frac{1 - F(\theta_i)}{f(\theta_i)} \right].
\]

Define
\[
J_i(x_i, \theta_i) = \psi(x_i - \theta_i) + \psi'(x_i - \theta_i) \frac{1 - F(\theta_i)}{f(\theta_i)}.
\]

We assume \( J_i \) is concave in \( x_i \), has an interior maximum over \( x_i \) for all \( \theta_i \), and \( \frac{\partial^2 J_i}{\partial x_i \partial \theta_i} \geq 0 \). Following the arguments similar to those in part (b), maximizing the principal’s objective function pointwise yields the following solution: choose \( x_i(\theta_i) \) to satisfy
\[
1 - \psi'(x_i(\theta_i) - \theta_i) = \psi''(x_i(\theta_i) - \theta_i) \frac{1 - F(\theta_i)}{f(\theta_i)},
\]

8
and choose $\phi_i = 1$ if $J_i > J_j$ for all $j \neq i$ and $\phi_i = 0$ if $J_j > J_i$ for some $j \neq i$. (You can specify any convenient rule to solve ties.) Under our assumptions on $J_i$, it is straightforward to check the monotonicity condition is satisfied.

5 Priority Contracts (Optional)

(a) Before proceeding, it is simplest to set up the following standarization:

$$q_i(\theta_i) \equiv E_{\theta_i}[q_i(\theta)], \quad t_i(\theta_i) \equiv E_{\theta_i}[t_i(\theta)], \quad U_i(\hat{\theta}_i|\theta_i) \equiv \theta_i q_i(\hat{\theta}_i) - t_i(\hat{\theta}_i),$$

and

$$U_i(\theta_i) \equiv U_i(\hat{\theta}_i|\theta_i).$$

Thus, the firm’s maximization program can be succinctly stated as

$$\max_{q_i \in \{0, 1\}, U_i} E_{\theta_i} \left[ \sum_{i=1}^{N} \theta_i q_i(\theta) - c q_i(\theta) - U_i(\theta) \right],$$

subject to (Bayesian Nash) incentive compatibility

$$U_i(\theta_i) \geq U_i(\hat{\theta}_i|\theta_i), \quad \forall \theta_i, \hat{\theta}_i \in [0, 1],$$

(Bayesian Nash) individual rationality

$$U_i(\theta_i) \geq 0, \quad \forall \theta_i \in [0, 1],$$

and available capacity

$$\sum_{i=1}^{N} q_i(\theta) \leq K.$$

(b) A mechanism is IC iff

$$\frac{dU_i(\theta_i)}{d\theta_i} = q_i(\theta_i)$$

and $q_i(\theta_i)$ is nondecreasing.

(c) Incorporating the IC and IR constraints into the objective function can be accomplished by integrating by parts:

$$E_{\theta}[U_i(\theta)] = E_{\theta}[q_i(\theta) 1 - \frac{F(\theta_i)}{f(\theta_i)}].$$

Substituting this into the objective function yields the simpler program

$$\max_{q_i \in \{0, 1\}} E_{\theta} \left[ \sum_{i=1}^{N} \left( \theta_i - c - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i(\theta) \right],$$

subject to $\sum_{i=1}^{N} q_i(\theta) \leq K$ and $q_i$ monotonicity. Forming the Lagrangian (ignoring for now the monotonicity constraints), we obtain

$$L = E_{\theta} \left[ \sum_{i=1}^{N} \left( \theta_i - c - \frac{1 - F(\theta_i)}{f(\theta_i)} \right) q_i(\theta) + \lambda(\theta) \left( K - \sum_{i=1}^{N} q_i(\theta) \right) \right],$$

where

$$\lambda(\theta) = \begin{cases} 1 & \text{if } \theta_i > \theta_j \text{ for all } j \neq i, \\ 0 & \text{if } \theta_j > \theta_i \text{ for some } j \neq i. \end{cases}$$
where the Lagrange multiplier on the capacity constraint is $\lambda(\theta)$. Define

$$J(\theta_i) \equiv \theta_i - c - \frac{1 - F(\theta_i)}{f(\theta_i)}.$$ 

The optimal auction will award a unit of output to every customer whose “Virtual” valuation is nonnegative, i.e., $J(\theta_i) - \lambda(\theta) \geq 0$. $\lambda(\theta)$ is endogenously determined as the minimum value of the multiplier which causes the capacity constraint to become slack. The monotonicity condition is satisfied given the MHR condition.

(d) For $K \geq N$, the capacity constraint is always slack, and so $\lambda(\theta) = 0$. This implies that we are in the standard multi-good auction case which is analogous to Myerson (1981). Here, the firm will award a unit to every customer whose type is such that $J(\theta_i) \geq 0$. Define $p^*$ as the unique solution to $J(p^*) = 0$. The optimal mechanism can be implemented by charging a price of $p^*$ (which does not depend upon $\theta$) to everyone; those with valuations above $p^*$ will purchase the good and those with lower valuations will receive nothing. You can come to this conclusion in two ways. First, you can solve for the optimal transfers and then show that they have the nature of a fixed price. Second, and far more easy, you can use the revenue equivalence result: any mechanism which implements the same allocation $\{q_i\}$ and the same reservation utility $U_i(0)$ will generate the same expected revenue. Here, a threshold price of $p^*$ will result in purchases iff $J(\theta_i) \geq 0$ — the same as the direct mechanism. Additionally, anyone with a valuation below $p^*$ will not purchase and instead receive $U_i(0) = 0$.

(e) When $K < N$, it is possible that the capacity constraint will bind. Thus, $\lambda(\theta) > 0$ for some values of $\theta$. Note that $\lambda(\theta)$ is the minimal value of the multiplier such that the capacity constraint is satisfied. Define $\theta_m(\theta)$ to be the value of the m-th highest $\theta_i$. Then $\lambda(\theta)$ will be equal to the larger of 0 and $J(\theta_m(\theta))$. For a given $\lambda$, the price that a consuming customer pays is given by $J(p^*_\lambda) = \lambda$. MHRC implies that the higher the shadow cost of capacity, the more that consumers will pay for consumption. As an aside, because $\lambda(\theta) > 0$ iff $J(\theta_{K+1}(\theta)) \geq 0$, we can define $p^*$ as the solution to

$$J(p^*) = \max\{0, J(\theta_{K+1}(\theta))\}$$

for the general case. Here, customers announce their type, and the price $p^*(\theta)$ is posted at which customers can purchase. Clearly, in the indirect auction the capacity constraint is satisfied and the good is allocated to the highest (non-negative) virtual-type customers. (It is straightforward to show that in a standard auction, bidding functions are monotonic.) Because the reservation price can be set so that the lowest type earns no rents ($J(p^*) = 0$), we can apply the revenue equivalence theorem to verify the desired result. Thus, we know that this auction yields the same expected value as any indirect mechanism.