1 Fudenberg-Holmstrom-Milgrom (9 pts.)

(a) The only difference of this model from Fudenberg-Holmstrom-Milgrom paper is that the agent’s actions affect the principal’s outside options.

Let $\Delta$ be an optimal long-term contract. We can easily check that all the assumptions in Theorem 2 is satisfied, and thus there exists a sequentially efficient long-term contract $\Delta'$ which is equivalent to $\Delta$. (Note that Theorem 2 and its proof don’t depend on the principal’s outside options.)

Next, we want to find a sequentially optimal contract $\hat{\Delta}$ which is equivalent to $\Delta'$ as the paper does in Theorem 3, but we cannot directly apply Theorem 3 because the theorem relies on the fact that the principal’s outside option is always zero.

However, we can construct a sequentially optimal contract $\hat{\Delta}$, by essentially the same idea as Theorem 3, as following.

First note that the sequentially efficient contract $\Delta'$ can be contingent not only on $x$ but also on $c$ and $y$ because these are also contractible. Therefore the payment $s_t'$ is a function of $h^t = (c^t, x^t, y^t)$.

Now we can transfer payment from period to period by setting

$$\hat{s}_t(h^t) = s_t'(h^t) + (\Pi_{t}(\Delta' \setminus h^{t-1}) + y_{t-1}) - (\Pi_{t+1}(\Delta' \setminus h^t) + y_t).$$

By the same argument as Theorem 3, this contract is sequentially optimal.

(b) Since there is no moral hazard problem here (all actions are observable), an optimal long-term contract achieves the first best. To see this, the first best is given by

$$\max_{s_t, c_t, x_t, y_t} \sum_{t+1}^T U(c_t, x_t, y_t),$$

subject to

$$(IR) \sum_t s_t = \sum_t x_t, (BB) \sum_t c_t = \sum_t s_t.$$
By combining two constraints we get \( \sum_t c_t = \sum_t x_t \), and we can solve for the optimal \( c, x \) and \( y \). The first best solution is given by \( y^*_t = 0 \) for all \( t \), \( c^*_t = c^* = x^*_t = x^* \) for all \( t \), where \( c^* = x^* \) is the solution to

\[
U_c(c, x, 0) + U_x(c, x, 0) = 0.
\]

Effort levels and consumption levels are smoothed over time, and the marginal utility of consumption is equal to marginal cost of effort.

This first best solution is implementable by the payment rule \( s_t = x_t \) for all \( t \) and for all history \( h^t \), because now the agent’s problem is to maximize his payoff subject to (BB) \( \sum_t c_t = \sum_t s_t \), which is equivalent to the problem above.

(c) Consider a history \( h^1 = (c^*, x^*, y_1) \), where \( y_1 > 0 \). It is easy to check that under the long-term contract in part (b), the agent will choose \( c^*_t = c^* = x^*_t = x^* \) in the remaining periods after \( h^1 \). Therefore the principal’s continuation payoff is \( \Pi(\Delta \setminus h^1) = 0 \). However, this is not equal to her outside option \(-y_0\). Therefore this contract is not sequentially optimal.

2 Dynamic Adverse Selection (6 pts.)

(a) There are potentially two kinds of equilibria: “Separating equilibria,” where high type and low type sign different contracts, and “pooling equilibria,” where both types sign the same contract.

First consider separating equilibria. The contract offered to the low type is obviously “no contract,” that is, the wage is zero for all possible outcome (low outcome). Thus we are interested in characterizing the contract offered to the high type. Assuming the agent has full bargaining power, the problem is to maximize the agent’s expected utility, subject to (IC) low type doesn’t want to take this contract, (IR) principal get zero expected payoff, and no bidding constraint.

Let \( x > 0 \) be the high outcome. Let \( w_i, i = L, H \) be the first period wage when the first period outcome is \( i \), and \( w_{ij}, i, j = L, H \) be the second period wage when the first period outcome is \( i \) and the second period outcome is \( j \). Let \( u \) be the agent’s utility function and normalize \( u(0) = 0 \). Then the IC constraint is

\[
u(w_L) + u(w_{LL}) \leq 0,
\]

and the IR constraint is

\[
p_H(w_H + p_H w_{HH} + (1-p_H) w_{HL}) + (1-p_H)(w_L + p_H w_{LH} + (1-p_H) w_{LL}) = 2p_H x.
\]

To consider no bidding constraint, if the market has observed high outcome in the first period, then the market knows the agent is high type for sure, and thus offers him his expected productivity \( p_H x \). Thus no bidding constraint is given by

\[
p_H u(w_{HH}) + (1-p_H) u(w_{HL}) \geq u(p_H x).
\]
To solve this problem, first we ignore the no bidding constraint and then check that the solution actually satisfies the no bidding constraint. With only IC and IR constraint, we want to smooth the agent’s consumption as much as possible. Thus the solution is

\[ w_L = w_{LL} = 0 \quad \text{and} \quad w_H = w_{HH} = w_{HL} = w_{LH} = \frac{2x}{3 - p_H}. \]

(I’m being sloppy, but you can check it rigorously.) No bidding constraint is satisfied because

\[ \frac{2x}{3 - p_H} > p_H x \quad \forall p_H \in (0, 1). \]

Next consider pooling equilibria. In fact, it is a pooling equilibrium that both types of agents signs the contract for high type described above. The firm (principal) is getting zero profit conditional on the agent being high type and also zero profit conditional on the agent being low type, and thus getting zero ex ante expected profit. Also, there is no contract which gives either type a higher expected payoff than this contract, given the firm getting zero profit.

(b) The pooling equilibrium in part (a) is still an equilibrium, because no information is revealed from the fact the agent has signed the contract. However, there is no separating equilibrium any more. If there is a separating equilibrium, the market knows that the agent is high type only from the fact that the agent has signed the contract for high type, and the market is willing to bid the agent up to \( p_H x \). Then, the low type has an incentive to sign the high type contract not for the wage from the contract itself, but for the market wage in the second period.

3 Regulation (6 pts.)

(a) Since the agent is risk neutral, we can achieve the first best by “selling the firm” to the agent; i.e., setting the revenue function so that the agent’s payoff maximization problem exactly coincides with social surplus maximization.

The first best is to solve

\[ \max_{x_1, x_2, I} px_1 - c(x_1, I) + px_2 - c(x_2, I) - kI, \]

and the solution is given by

\[ x_1^* = x_2^* = x^*, \]
\[ p = c_x(x^*, I^*), \]
\[ -2c_I(x^*, I^*) = k. \]

The regulator’s optimal strategy is to choose

\[ R^1(x_1) = px_1 + \beta, \quad R^2(x_2) = px_2, \]

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where $\beta$ is a constant. Then the agent’s problem is to solve
\[
\max_{x_1, x_2, I} R^1(x_1) - c(x_1, I) + R^2(x_2) - c(x_2, I) - kI,
\]
and the solution is same as the first best. The constant $\beta$ should be chosen so that the agent’s IR constraint is binding.

(b) Suppose there exists a pure strategy sequential equilibrium, and suppose on the equilibrium path, the regulator offers contracts $R^1$, $R^2$ and the firm chooses $I^* > 0$, $x_1^*$ and $x_2^*$.

In the second period, the regulator knows (correctly expects) $I^*$ and offers a contract $R^2$ such that the firm is indifferent between quitting and not quitting, and the firm doesn’t quit.

One necessary condition is, $I^*$ must maximize the firm’s total profit, conditionally on choosing $x_1^*$ and $x_2^*$ and not quitting in either period. That is,
\[
\frac{\partial}{\partial I} \left( R^1(x_1^*) - c(x_1^*, I^*) - kI + R^2(x_2^*) - c(x_2^*, I^*) \right) = 0 \quad (1)
\]
\[
k = -c_I(x_1^*, I^*) - c_I(x_2^*, I^*). \quad (2)
\]
and therefore
\[
k > -c_I(x_1^*, I^*). \quad (3)
\]

Since the second period profit is zero in equilibrium, the firm is in fact receiving a payoff of
\[
R^1(x_1^*) - c(x_1^*, I^*) - kI^*.
\]
If the firm decrease $I$ by a small amount $\Delta I$, i.e., choose $I = I^* - \Delta I$ and quit production in the second period, then the second period profit is unchanged and the first period profit changes by
\[
c_I(x_1^*, I^*) \Delta I + k \Delta I.
\]
Because of (3), this is positive, i.e., the firm gains by decreasing $I$. This means that the original choice of $I$ doesn’t constitute an equilibrium.

4 Aggregation and Linearity (Optional)

(a) Define $u$ by $u(x) = -\exp(-rx)$. Notice that the exponential utility has the following property.
\[
u(x + y) = -\exp(-r(x + y)) = -\exp(-rx)\exp(-ry) = -u(x)u(y). \quad (4)
\]
Let $\{s_{ij}\}_{i,j=1,2}$ be a sharing rule where $\{s_{ij}\}$ is the wage when the first period outcome is $i$ and the second period outcome is $j$. For any sharing rule $\{s_{ij}\}_{i,j=1,2}$, define $\{w_i\}_{i=1,2}$ by
\[
u(w_i) = qu(s_{i1} - c_1) + (1 - q)u(s_{i0} - c_1).
\]
This means that after any effort level and output $i$ in period 1, the agent is indifferent between (i) putting effort level $e_2 = 1$ and receiving a wage $s_{i1}$ or $s_{i0}$ depending on the second period output, and (ii) putting effort level $e_2 = 0$ and receiving a fixed wage $w_i$ in period 2.

Since we are assuming that it is optimal to have the agent work in both periods, the principal’s objective is to minimize the expected wage payment under IC and IR constraints.

The problem can be written as

$$\min_{\{s_{ij}\}, i, j = 1, 2} q(s_{i1} + (1 - q)s_{i0}) + (1 - q)(qs_{01} + (1 - q)s_{00})$$

s.t. (IC1) $q(u(s_{i1} - 2c_1) + (1 - q)u(s_{i0} - 2c_1))$

$$+ (1 - q)(qu(s_{01} - 2c_1) + (1 - q)u(s_{00} - 2c_1))$$

$$\geq p(qu(s_{11} - c_1) + (1 - q)u(s_{10} - c_1))$$

$$+ (1 - p)(qu(s_{01} - c_1) + (1 - q)u(s_{00} - c_1))$$,

(II) $qu(s_{i1} - 2c_1) + (1 - q)u(s_{i0} - 2c_1)$  $i = 1, 2,$

(IIR) $q(u(s_{i1} - 2c_1) + (1 - q)u(s_{i0} - 2c_1))$

$$+ (1 - q)(qu(s_{01} - 2c_1) + (1 - q)u(s_{00} - 2c_1)) \geq u(0).$$

(IC1) says that it is better to choose $e_1 = 1$ and $e_2 = 1$ than $e_1 = 0$ and $e_2 = 1$. (IC2) says that if the agent has chosen $e_1 = 0$ and the output $i$ has realized in period 1, then it is better to choose $e_2 = 1$ than $e_2 = 0$. (IR) has the usual meaning.

You may think that we need to consider other deviations such as $e_1 = 0$ and $e_2 = 0$, but it turns out the IC constraints described above are actually sufficient by the following argument.

Using the formula (1), (IC2) can be rewritten as

$$\text{(IC2') } qu(s_{i1} - c_1) + (1 - q)u(s_{i0} - c_1) \geq pu(s_{i1}) + (1 - p)u(s_{i0}) \ i = 1, 2,$$

This means that even if the agent has chosen $e_1 = 0$ in period 1, the agent still wants to choose $e_2 = 1$ after any output in period 1. Therefore (IC1) and (IC2) are the necessary and sufficient incentive constraints.

Using again the formula (1) and from the definition of $w_i$, (IC1) can be rewritten as

$$\text{(IC1’) } qu(w_i - c_1) + (1 - q)u(w_0 - c_1) \geq p(u_i) + (1 - p)u(w_0) \ i = 1, 2.$$

The optimal long-term contract problem can be solved in the following procedure.

1. For each $i$, fix $w_i$ and choose $s_{i1}$ and $s_{i0}$ to minimize $qs_{i1} + (1 - q)s_{i0}$ subject to (IC2') the agent has an incentive to choose $e_1 = 1$ and (IR2) the agent receives a certainty equivalent greater than or equal to $w_i$. Formally,

$$\text{(IR2) } qu(s_{i1} - c_1) + (1 - q)u(s_{i0} - c_1) = u(w_i).$$
2. This tells you the minimum expected cost to give the agent a certainty equivalent of \( w_i \), namely \( d(w_i) \).

3. Choose \( w_i \) to minimize \( qd(w_1) + (1 - q)d(w_0) \) subject to (IC1') the agent has an incentive to choose \( e_1 = 1 \) and (IR1) the agent receives a certainty equivalent greater than or equal to 0. Formally,

\[
\text{(IR2)} \quad qu(w_1 - c_1) + (1 - q)u(w_0 - c_1) = u(0).
\]

Now we solve the model. First, we look at the second period problem, and (IC2') can be rewritten as

\[
\text{(IC2')} \quad qu(s_{i1} - s_{i0} - c_1) + (1 - q)u(-c_1) \geq pu(s_{i1} - s_{i0}) + (1 - p)u(0) \quad i = 1, 2.
\]

This is equivalent to

\[
s_{i1} - s_{i0} \geq \bar{s}
\]

for some \( \bar{s} \). The IC constraint says that the wage difference between success and failure must be greater than equal to \( \bar{s} \). This constraint is binding because since the agent is risk averse and the principal is risk neutral, the wage difference must be as small as possible. Also notice that \( \bar{s} \) is determined only from (IC2') and therefore it doesn’t depend on the level of the reservation certainty equivalent \( w_i \). Therefore,

\[
s_{11} - s_{10} = s_{01} - s_{00} = \bar{s}.
\]

The next observation is that since the wage difference is fixed, if \( w_i \) is increased by \( \Delta \), then you should just increase \( s_{i1} \) and \( s_{i0} \) by \( \Delta \). This increases expected wage payment by \( \Delta \).

Therefore, if we look at the first period problem, the objective of minimizing the expected wage payment is simply equal to minimizing the expected value of \( w_i \), i.e., \( qw_1 + (1 - q)w_0 \).

Then the first period problem is equivalent to the second period problem, where \( s_{i1} \) and \( s_{i0} \) are replaced by \( w_1 \) and \( w_0 \), and \( w_i \) is replaced by 0. Therefore, we have

\[
w_1 - w_0 = s_{11} - s_{10} = s_{01} - s_{00} = \bar{s}.
\]

This is going to be true if and only if

\[
s_{11} - s_{10} = s_{01} - s_{00} = \bar{s} \quad \text{and} \quad s_{10} = s_{01},
\]

and thus the incentive scheme is linear.

(b) The problem is equivalent to a static problem, where the agent chooses effort levels for both periods at the same time. Now we have three incentive constraints: choosing \((e_1, e_2) = (1, 1)\) must be better than choosing \((1, 0), (0, 1)\) and \((0, 0)\).

In part (a), we didn’t actually solve for the optimal contract, but we only showed that the principal’s problem can be decomposed into three problems,
which are essentially equivalent. However, to show the optimal contract is not linear here, we need to actually solve for the optimal contract. This makes the problem hard.