1 Nash Implementation

(a) First, we show that condition 1 is satisfied. Suppose, to the contrary, \( y = (x, t_1, \cdots, t_I) \in f(\theta), \) \( x = 1 \) but \( y \notin f(\phi) \), where \( \phi \geq \theta \). Monotonicity implies there exist a player \( i \) and an outcome \( \tilde{y} = (\tilde{x}, \tilde{t}_1, \cdots, \tilde{t}_I) \) such that

\[ y \succeq^\theta_i \tilde{y}, \quad \tilde{y} \succ^\phi_i y, \]  

where \( \succ^\theta_i \) represents player \( i \)'s preference under state \( \theta \). These can be rewritten as

\[ \theta^i x + t^i \geq \theta^i \tilde{x} + \tilde{t}^i, \]  
\[ \phi^i \tilde{x} + \tilde{t}^i > \phi^i x + t^i. \]  

(4) is equivalent to

\[ -\phi^i x - t^i > -\phi^i \tilde{x} - \tilde{t}^i. \]  

Adding up (3) and (5),

\[ (\theta^i - \phi^i)x > (\theta^i - \phi^i)\tilde{x}, \]  
\[ (\theta^i - \phi^i)(x - \tilde{x}) > 0. \]

Since \( \phi \geq \theta \), we need to have \( \phi > \theta \) and \( \tilde{x} > x \). But since \( x = 1 \), it must be \( \tilde{x} > 1 \), which is impossible. Contradiction.

We can prove condition 2 in the same way.

(b) Suppose \( f \) is not monotonic. Then there exist \( y = (x, t_1, \cdots, t_I), \theta, \) and \( \phi \) such that \( y \in f(\theta), y \notin f(\phi), \) and for all \( i \) and for all \( \tilde{y} = (\tilde{x}, \tilde{t}_1, \cdots, \tilde{t}_I), \)

\[ y \succeq^\theta_i \tilde{y} \implies y \succeq^\phi_i \tilde{y}. \]  

This is equivalent to
\[ \theta^i x + t^i \geq \tilde{\theta}^i \tilde{x} + \tilde{t}^i \implies \phi^i x + t^i \geq \tilde{\phi}^i \tilde{x} + \tilde{t}^i. \]  
(7)

If \( x = 1 \), take a \( \tilde{y} \) such that \( \tilde{x} = 0 \). Then, (7) is equivalent to
\[ \theta^i + t^i \geq \tilde{t}^i \implies \phi^i + t^i \geq \tilde{t}^i. \]

This implies \( \phi^i \geq \theta^i \) for all \( i \). Because we assumed \( y \in f(\theta), y \notin f(\phi) \), This contradicts condition 1.

If \( x = 0 \), take a \( \tilde{y} \) such that \( \tilde{x} = 1 \). Then, (7) is equivalent to
\[ t^i \geq \theta^i + \tilde{t}^i \implies t^i \geq \phi^i + \tilde{t}^i. \]

This implies \( \theta^i \geq \phi^i \) for all \( i \). Because we assumed \( y \in f(\theta), y \notin f(\phi) \), This contradicts condition 2.

\( c \)

Since the transfer \( t^i \) can be increased as much as you want, there isn’t “the most favored allocation” for any agent. Therefore no veto power condition is satisfied. From Maskin’s theorem, we can conclude \( f \) is Nash implementable.

\( d \)

Consider the following social choice rule: If \( \sum \theta^i \geq 0 \),
\[ f(\theta) = \{ y = (x, t^1, \cdots, t^I) | x = 1, t^i \leq \theta^i, \sum t^i = 0 \}. \]  
(8)

If \( \sum \theta^i \leq 0 \),
\[ f(\theta) = \{ y = (0, 0, \cdots, 0) \}. \]  
(9)

You can easily check \( f(\theta) \) is nonempty for all \( \theta \). It is also easy to see conditions 1 and 2 are satisfied. By the discussion from (a) to (c), \( f \) is nash implementable.

\( e \)

As we have seen in part (d), we can implement an efficient social choice correspondence. However, we can hardly say this social choice correspondence is “fair”. For example, suppose \( I = 100 \) and consider two states \( \theta = (\theta^1, \cdots, \theta^{100}) = (-98, 1, 1, \cdots, 1) \) and \( \phi = (\phi^1, \cdots, \phi^{100}) = (1, \cdots, 1) \). In both states it is efficient to choose \( x = 1 \) We may think that in a “fair” sharing of the cost agent 1 should be compensated a lot in state \( \theta \). However, since \( \phi \geq \theta \), condition 1 implies that this output should also included in \( f(\phi) \). We don’t think it is “fair” that agent 1 is compensated a lot in state \( \phi \).
2 Dominant Strategy versus Nash Implementation

(a) Prove by contradiction. Suppose a mechanism \((\{M^i\}_{i=1}^3, g)\) implements \(f\) in dominant strategy, where \(M^i\) is player \(i\)'s message space and \(g\) is the decision rule. If \(m^i_1 \in M^1\) is player \(i\)'s dominant strategy when the state is \((\theta^1_1, \theta^2_1)\), then it must also be her dominant strategy in state \((\theta^1_1, \theta^2_2)\), because whether a strategy dominates another doesn’t depend on other players’ preferences. Based on this, for \(i = 1, 2\) and \(k = 1, 2\), let \(m^i_k\) be player \(i\)'s dominant strategy when \(\theta^i = \theta^i_k\). Also, let \(m^3\) be player 3's dominant strategy, which is independent of state for the same reason. For this mechanism to implement \(f\), we need to have

\[
g(m^1_1, m^2_1, m^3) = a, \quad g(m^1_1, m^2_2, m^3) = d, \quad g(m^1_2, m^2_1, m^3) = d, \quad g(m^1_2, m^2_2, m^3) = b.
\]

Then, \(m^1_1\) is not a dominant strategy when \(\theta^1 = \theta^1_1\) because she does better by playing \(m^1_1\) if player 2 and 3 are choosing \(m^2_2\) and \(m^3\). Contradiction.

(b) Check each pair of states satisfies the condition in the definition of monotonicity.

(c) At least \(N - 1\) agents agree on the most preferred decision in states \((\theta^1_1, \theta^2_1)\) and \((\theta^1_2, \theta^2_2)\), and in these states their most preferred decision is indeed selected.

(d) From (b) and (c), Maskin’s theorem tells us \(f\) is Nash implementable.

3 Risk Sharing and Implementation

There is a typo in the problem. the assumption must be

\[v(0, \theta) = k(0, \theta) = 0 \forall \theta.\]

(a) We want to find the first best price and quantity contingent on state \(\theta\); \((p^*(\theta), q^*(\theta))\). First, after any realization of \(\theta\), the quantity must be ex post efficient, i.e.,

\[q^*(\theta) \in \arg\max v(q, \theta) - k(q, \theta).\]
We say state $\theta$ is higher than state $\theta'$ and denote $\theta > \theta'$ if 
$$v(q^*(\theta), \theta) - k(q^*(\theta), \theta) > v(q^*(\theta'), \theta') - k(q^*(\theta'), \theta').$$

Next, given $q^*(\theta)$, we want to find optimal $p$. Note that, because both agents are risk averse, ex ante expected utility possibility frontier is convex. So any first best $p$ maximizes weighted sum of ex ante expected payoffs of the buyer and the seller. That is, $p^*(\cdot)$ is the solution to
$$\max_{p(\cdot)} E[u_b(v(q^*(\theta), \theta) - p(\theta))] + \lambda E[u_s(p(\theta) - k(q^*(\theta), \theta))].$$

Or, equivalently,
$$\max_{p(\cdot)} \int_{\theta \in \Theta} (u_b(v(q^*(\theta), \theta) - p(\theta)) + \lambda u_s(p(\theta) - k(q^*(\theta), \theta))) f(\theta) d\theta.$$
So each $p(\theta)$ maximizes the integrand pointwise. The FOC is
$$u_b'(v(q^*(\theta), \theta) - p^*(\theta)) = \lambda u_s'(p^*(\theta) - k(q^*(\theta), \theta)),$$
or equivalently,
$$\frac{u_b'(v(q^*(\theta), \theta) - p^*(\theta))}{u_s'(p^*(\theta) - k(q^*(\theta), \theta))} = \lambda. \quad (10)$$

This implies that when $v(q^*(\theta), \theta) - k(q^*(\theta), \theta)$ increases, both $v(q^*(\theta), \theta) - p^*(\theta)$ and $p^*(\theta) - k(q^*(\theta), \theta)$ increase accordingly, and therefore both agents are better off. In other words, both agents are better off in higher state.

(b)

The following contract implements the first best.

- ask the two parties to announce the state of the nature.
- if the parties announce the same state $\theta$, renegotiation starts from the corresponding first best $(p^*(\theta), q^*(\theta))$.
- if the buyer announces a higher state than the seller, full bargaining power is granted to the buyer and renegotiation starts from $(p, q)$ where $q = 0$ and $p$ is such that
  $$u_s(p - 0) = u_s(p^*(\theta) - k(q^*(\theta), \theta)),$$
  where $\theta$ is the state announced by the seller.
- if the seller announces a higher state than the buyer, full bargaining power is granted to the seller and renegotiation starts from $(p, q)$ where $q = 0$ and $p$ is such that
  $$u_b(-p) = u_b(v(q^*(\theta), \theta) - p^*(\theta)),$$
  where $\theta$ is the state announced by the buyer.
In other words, if two parties announce different states, then the full bargaining power is granted to the optimistic party and the renegotiation starts from $(p, q)$, where $p = 0$ and $p$ gives the pessimistic party the first best utility level corresponding to the utility level he announced.

**Proof:**

It suffices to show that truth-telling is a Nash equilibrium. Consider a unilateral deviation from truth-telling, say by the buyer. If he announces a lower state $\tilde{\theta}$ when the true state is $\theta$, his payoff at the default option is equal to $u_b(v(q^*(\tilde{\theta}), \tilde{\theta}) - p^*(\tilde{\theta}))$, which is less than what he would have got without deviating, and since he has no bargaining power, he can not improve upon it. If he announces a higher state $\tilde{\theta}$ when the true state is $\theta$, then the seller has the default payoff $u_s(p^*(\theta) - k(q^*(\theta), \theta))$ and thus the buyer need to leave at least this payoff to the seller, and this payoff of the seller is exactly same as what she is getting if both players are telling the truth. Therefore, the buyer does not want to deviate.

(c)

Without a message game, the ex ante contract specifies a default contracting term $(p_0, q_0)$ which cannot be contingent on the state of the world.

After realization of $\theta$, the buyer and seller renegotiate over $p$ and $q$ taking $(p_0, q_0)$ as the disagreement point.

Let’s assume renegotiation with Nash bargaining solution. Then for any realization of $\theta$, the agreement $(p, q)$ solves

\[
\max \alpha \log(u_b(v(q, \theta) - p) - u_b(v(q_0, \theta) - p_0)) + (1 - \alpha) \log(u_s(p - k(q, \theta)) - u_s(p_0 - k(q_0, \theta))),
\]

where $\alpha \in [0, 1]$ is a parameter of bargaining power.

It is easy to see the solution has $q = q^*$ (the agreement must choose some ex post efficient allocation). Substituting $q = q^*$ and taking FOC with respect to $p$, we get

\[
\frac{-\alpha u_b'(v(q^*, \theta) - p)}{u_b(v(q^*, \theta) - p) - u_b(v(q_0, \theta) - p_0)} + \frac{(1 - \alpha)u'_s(p - k(q^*, \theta))}{u_s(p - k(q^*, \theta)) - u_s(p_0 - k(q_0, \theta))} = 0.
\]

This can be rewritten as

\[
\frac{u_b'(v(q^*, \theta) - p)}{u'_s(p - k(q^*, \theta))} = \frac{1 - \alpha}{\alpha} \frac{u_b(v(q^*, \theta) - p) - u_b(v(q_0, \theta) - p_0)}{u_s(p - k(q^*, \theta)) - u_s(p_0 - k(q_0, \theta))}.
\]

(11)

Compare this with the FOC for first best (10). To be able to implement the first best, RHS of (11) where $p = p^*(\theta)$ need to be equal to $\lambda$ for all $\theta$. But this is not generally true. Therefore, the first best cannot be implemented.

---

1There might be other Nash equilibria, but the final allocation are the same $(p^*(\theta), q^*(\theta))$ in all equilibria, because the message game is a constant sum game and and thus the equilibrium payoff must be the same.
4 Incomplete Contracts

(a) Since we have transferrable utilities, the first best is to maximize the sum of the buyer’s and the seller’s utilities, i.e.,

$$\max_{q,r} U^S + U^B = u(q,r) - cq - r.$$ 

Assuming $u$ is concave and we have interior solutions, the first best is characterized by

$$u_q(q^*, r^*) = c,$$

$$u_r(q^*, r^*) = 1.$$ 

(b) We solve for the equilibrium by backward induction. At $t = 3$, the seller accepts the renegotiation offer if and only if her payoff is higher than or equal to her payoff from $(q,t) = 0,0$, i.e.,

$$t - cq - r \geq -r.$$ 

At $t = 2$, the buyer makes a renegotiation offer which maximizes his payoff conditional on the offer being accepted, i.e., he solves

$$\max_{q,t} u(q,r) - t$$

s.t. $t - cq - r \geq -r.$

The solution must satisfy

$$u_q(q,r) = c,$$

$$t = cq.$$ 

At $t = 1$, the seller knows that her payoff will be $-r$ no matter what $r$ she chooses, so she optimally chooses $r = 0$.

The first best is not achieved.

(c) Suppose the initial contract specifies a quantity-transfer pair $(q_0,t_0)$. At $t = 3$, the seller accepts the renegotiation offer $(q,t)$ if and only if

$$t - cq - r \geq t_0 - cq_0 - r.$$
Therefore, at $t = 2$, the buyer’s renegotiation offer solves
\[
\max_{q,t} u(q,r) - t \\
\text{s.t. } t - cq - r \geq t_0 - cq_0 - r.
\]
The solution must satisfy
\[
\begin{align*}
u_q(q,r) &= c, \\
t &= cq + (t_0 - cq_0).
\end{align*}
\]
At $t = 1$, the seller knows that her payoff will be $t_0 - cq_0 - r$ no matter what $r$ she chooses, so she optimally chooses $r = 0$.

The first best is not achieved.

(d)
Suppose the initial contract specifies a quantity-transfer pair $(q_0,t_0)$. At $t = 3$, if the seller rejects buyer’s renegotiation offer, the buyer chooses $(0,0)$ or $(q_0,t_0)$ which gives him higher payoff. That is, he chooses $(q_0,t_0)$ if
\[
u(q_0,r) - t_0 \geq u(0,r)
\]
and chooses $(0,0)$ otherwise. Therefore the seller’s payoff when the renegotiation offer is rejected is $U^S = t_0 - cq_0 - r$ if $u(q_0,r) - t_0 \geq u(0,r)$ and $U^S = -r$ otherwise. Let’s denote this $U^s(t_0,q_0,r)$. At $t = 3$, the seller accepts the buyer’s renegotiation offer if and only if the payoff from accepting the offer is at least $U^s(t_0,q_0,r)$.

At $t = 2$, the buyer’s renegotiation offer solves
\[
\max_{q,t} u(q,r) - t \\
\text{s.t. } t - cq - r \geq U^s(t_0,q_0,r).
\]
The solution must satisfy
\[
\begin{align*}
u_q(q,r) &= c, \\
t &= cq + r + U^s(t_0,q_0,r).
\end{align*}
\]
At $t = 1$, the seller knows that her payoff will be $U^s(t_0,q_0,r)$ no matter what $r$ she chooses, so she chooses $r$ to maximize the value of $U^s(t_0,q_0,r)$. Specifically, she can either choose $r = 0$ and receive payoff $U^s(t_0,q_0,r) = 0$ or choose minimum necessary $r$ with which the buyer picks $(q_0,t_0)$ in case renegotiation breaks down and receive payoff $U^s(t_0,q_0,r) = t_0 - cq_0 - r$. In other words, let $\tilde{r}$ the solution to
\[
u(q_0,r) - t_0 = u(0,r),
\]
and
and the seller chooses \( r = 0 \) if \( t_0 - cq_0 - r < 0 \) and chooses \( r = \bar{r} \) if \( t_0 - cq_0 - r \geq 0 \).

If the initial contract \((q_0, t_0)\) is such that \( u(q_0, r^*) - t_0 = u(0, r^*) \) and \( t_0 - cq_0 - r^* \geq 0 \), then the seller chooses \( r^* \) and the buyer chooses \( q^* \), i.e., the first best is achieved.

(e)

Now the first best is characterized by

\[
\max_{q,r} u(q) - c(r)q - r,
\]

\[
u'(q^*) = c(r^*),
\]

\[
c'(r^*)q^* = -r^*.
\]

At \( t = 2 \), the buyer’s renegotiation offer solves

\[
\max_{q,t} u(q) - t
\]

\[
\text{s.t. } t - c(r)q - r \geq t_0 - c(r)q_0 - r.
\]

and the solution is given by

\[
u'(q) = c(r),
\]

\[t = t_0 + c(r)(q - q_0).
\]

Therefore, at \( t = 1 \), the seller solves

\[
max_{r,t} t_0 - c(r)q_0 - r
\]

and thus she chooses \( r \) such that

\[
c'(r)q_0 = -r.
\]

Therefore, if the initial contracts specifies \( q_0 = q^* \), the first best is achieved.

5 Advocates

(a)

Part (a) is very complicated and the complete answer will be very very long. I won’t do it here, but rather I just solve the problem assuming the following, to illustrate the logic. (Whether these assumptions are actually satisfied or not depends on parameter values.)

- If no effort is made \((n = 0)\), the optimal decision is to choose \( d = 0 \).
• If an effort is made only in looking for evidence for A, the optimal decision is to choose $d = A$ if an evidence is observed and $d = 0$ otherwise. (Symmetric for the case an effort is made only for B.)

We calculate the expected loss for $n = 0, 1, 2$.

First consider the case $n = 2$. The loss occurs in the following case.

• Inertia happens when $\theta_A = -1$, $\theta_B = 0$ and an evidence for A is not observed. This happens with probability $\alpha(1 - \alpha)(1 - q)$.

• Inertia happens when $\theta_A = 0$, $\theta_B = 1$ and an evidence for B is not observed. This happens with probability $\alpha(1 - \alpha)(1 - q)$.

• Extermism happens when $\theta_A = -1$, $\theta_B = 1$ and only one evidence is observed. This happens with probability $2\alpha^2q(1 - q)$.

Therefore the expected loss is

$$EL_2 = 2\alpha(1 - \alpha)(1 - q)L_I + 2\alpha^2q(1 - q)L_E.$$ 

Second consider the case $n = 1$, where an effort is made only in A. The loss occurs in the following case.

• Inertia happens when $\theta_A = -1$, $\theta_B = 0$ and an evidence for A is not observed. This happens with probability $\alpha(1 - \alpha)(1 - q)$.

• Inertia happens when $\theta_A = 0$, $\theta_B = 1$. This happens with probability $\alpha(1 - \alpha)$.

• Extermism happens when $\theta_A = -1$, $\theta_B = 1$ and an evidence for A is found. This happens with probability $\alpha^2q$.

Therefore the expected loss is

$$EL_1 = \alpha(1 - \alpha)(2 - q)L_I + \alpha^2qL_E.$$ 

Lastly consider the case $n = 0$. Inertia happens when $\theta = -1$ or $\theta = 1$. This happens with probability $2\alpha(1 - \alpha)$. Therefore the expected loss is

$$EL_0 = 2\alpha(1 - \alpha)L_I.$$ 

It is optimal to have two units of effort expended if $EL_0 - EL_2 \geq 2K$ and $EL_0 - EL_1 \geq K$.

(b)
First consider the case of two agents. Each of two agents is in charge of the signal for one of $\theta_A$ and $\theta_B$.

The optimal contract for an agent is independent from the other agent’s performance (signal) because the error term is not correlated. (This is the “sufficient static result” we learned in the moral hazard part in 14.124.) Therefore the wage should depend only on whether the agent has provided information or not. Let $w_1$ be the wage when the signal is provided and $w_0$ be the wage when not. If the agent exert an effort at cost $K$, then a signal is provided with probability $x = \alpha q$.

The optimal (expected wage minimizing) contract to achieve the first best (implementing effort) is given by

$$
\min xw_1 + (1-x)w_0 \\
\text{s.t. (IC)} \ xw_1 + (1-x)w_0 - K \geq w_0 \\
\text{(IR)} \ xw_1 + (1-x)w_0 - K \geq 0.
$$

The optimal is obtained by $w_0 = 0$ and $w_1 = \frac{K}{2}$. Check that both constraints hold and we cannot decrease expected wage payment any more because (IR) is binding. The expected wage payment is $K$.

If we have limited liability, then the (IR) constraint is replaced by

$$
\text{(LL)} \ w_0, w_1 \geq 0.
$$

The solution is again $w_0 = 0$, $w_1 = \frac{K}{2}$ and the expected wage payment is $K$.

By offering this contract to both of the two agents, we can implement first best (implementing both effort) with an expected wage payment of $2K$ no matter if we have limited liability or not.

If we have one agent who is in charge of the signals for both states, let $w_{11}$, $w_{10}$, $w_{01}$ and $w_{00}$ be the wages for providing both signals, only signal for $\theta_A$, only signal for $\theta_B$, and no signal, respectively. The optimal contract to achieve the first best is given by

$$
\min x^2w_{11} + x(1-x)(w_{10} + w_{01}) + (1-x)^2w_{00} \\
\text{s.t. (IC1)} \ x^2w_{11} + x(1-x)(w_{10} + w_{01}) + (1-x)^2w_{00} - 2K \\
\geq xw_{10} + (1-x)w_{00} - K \\
\hspace{1cm} \text{(IC2)} \ x^2w_{11} + x(1-x)(w_{10} + w_{01}) + (1-x)^2w_{00} - 2K \\
\geq xw_{01} + (1-x)w_{00} - K \\
\hspace{1cm} \text{(IC3)} \ x^2w_{11} + x(1-x)(w_{10} + w_{01}) + (1-x)^2w_{00} - 2K \geq w_{00} \\
\text{(IR)} \ x^2w_{11} + x(1-x)(w_{10} + w_{01}) + (1-x)^2w_{00} - 2K \geq 0
$$

(IC1), (IC2) and (IC3) say that exerting effort in both way is better than effort only in $\theta_A$, effort in $\theta_B$, and no effort respectively.
Like in the case of two agents, the optimum obtained by \( w_{10} = w_{01} = w_{00} = 0, \ w_{11} = \frac{2K}{x^2} \) and the expected wage payment is \( 2K \).

If we have limited liability, then the (IR) constraint is replaced by

\[ \text{(LL)} \quad w_0, \ w_1 \geq 0. \]

Observe that in the optimal contract \( w_{10} = w_{01} = w_{00} = 0 \), because otherwise we can lower each one of them and decrease the expected wage payment without violating any constraints. Then the IC constraints are reduced to

\[ x^2w_{11} \geq 2K, \]

and thus the solution is \( w_{11} = \frac{2K}{x^2} \). With this contract, the expected wage payment is \( 2K \).

To conclude, having one agent or two doesn’t make any difference and the first best can be achieved in both cases. Limited liability doesn’t affect this result.

(c)

First consider the case of two agents. Suppose agent A is in charge of signal for state \( \theta_A \) and agent B is in charge of signal for state \( \theta_B \).

Consider the optimal contract for agent A. Let \( w_d \) be the wage when decision \( d \) is implemented. If efforts for both directions are made, \( d = A \) is implemented when only signal for \( \theta_A = -1 \) is observed, \( d = B \) is implemented when only signal for \( \theta_B = 1 \) is observed, and \( d = 0 \) is implemented when both signals are observed and when no signal is observed. The expected wage in this case is

\[ x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 \]

because \( d = A \) and \( d = B \) are implemented with probability \( x(1-x) \) and \( d = 0 \) is implemented with the remaining probability. If agent deviates and doesn’t exert an effort, then \( d = B \) is implemented (when signal for \( \theta_B = 1 \) is observed) with probability \( x \) and \( d = 0 \) is implemented with probability \( 1-x \).

Thus the optimal contract is given by

\[
\begin{align*}
\text{min} & \quad x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 \\
\text{s.t.} & \quad (\text{IC}) \quad x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - K \\
& \quad \geq (1-x)w_0 + xw_B \\
& \quad (\text{IR}) \quad x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - K \geq 0.
\end{align*}
\]

The optimal is obtained by \( w_0 = w_B = 0 \) and \( w_A = \frac{K}{x(1-x)} \). Check that both constraints hold and we cannot decrease expected wage because (IR) is binding. The expected wage payment is \( K \).
If we have limited liability, (IR) is replaced by

\[ (LL) \quad w_A, w_B, w_0 \geq 0, \]

and again the solution is \( w_0 = w_B = 0 \) and \( w_A = \frac{K}{x(1-x)} \), with which the expected wage payment is \( K \).

The optimal contract problem for agent B is the same as for agent A. Therefore, no matter whether we have limited liability or not, first best can be achieved with the total expected payment of \( 2K \).

Next consider the case of one agent. The expected wage is

\[ x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 \]

again and now there are three IC constraints;

1. (IC1) \( x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - 2K \geq w_0 \)
2. (IC2) \( x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - 2K \geq xw_A + (1-x)w_0 - K \)
3. (IC3) \( x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - 2K \geq xw_B + (1-x)w_0 - K \).

(IC1), (IC2) and (IC3) says that exerting effort in both direction is better than no effort, only effort for \( \theta_A \), and only effort for \( \theta_B \), respectively.

(IC2) and (IC3) can be rewritten as

\[ x(1-x)(w_A + w_B) + (1-2x(1-x))w_0 - 2K \geq x \max\{w_A, w_B\} + (1-x)w_0 - K. \]

If \( w_A > w_B \), then by decreasing \( w_A \) and increasing \( w_B \) by the same small amount, we can relax this constraint without changing the expected wage payment, and thus without violating any other constraints. Therefore, at the optimum, it must be \( w_A = w_B = w \).

Now the expected wage can be rewritten as

\[ w_0 + 2x(1-x)(w-w_0), \]

and IC constraints can be reduced to

1. (IC1') \( x(1-x)(w-w_0) \geq K \)
2. (IC2') \( x(1-2x)(w-w_0) \geq K \)

From (IC1'), \( w > w_0 \) must hold because \( K > 0 \).

If \( x \geq 1/2 \), \( w > w_0 \) implies LHS of (IC2') is less than or equal to 0, and (IC2') cannot hold. First best cannot be achieved when \( x \geq 1/2 \).

If \( x < 1/2 \), (IC2') implies (IC1') and thus we can ignore (IC1').
Summarizing what we have done, the cost minimization problem when $x < 1/2$ can be reduced to

$$\min w_0 + 2x(1 - x)(w - w_0)$$

s.t. (IC) $x(1 - 2x)(w - w_0) \geq K$

(IR) $w_0 + 2x(1 - x)(w - w_0) - 2K \geq 0$

Letting $\Delta w = w - w_0$, this is rewritten as

$$\min w_0 + 2x(1 - x)\Delta w$$

s.t. (IC) $x(1 - 2x)\Delta w \geq K$

(IR) $w_0 + 2x(1 - x)\Delta w - 2K \geq 0$

If we let

$$\Delta w = \frac{K}{x(1 - 2x)}$$

and choose $w_0$ so that (IR) is binding, we can implement the first best with expected wage payment of $2K$.

If we have limited liability, (IR) is replaced by

(LL) $w_0 \geq 0$.

Now the optimal contract is

$$w_0 = 0, \ w = \Delta w = \frac{K}{x(1 - 2x)},$$

and the expected wage payment is

$$2x(1 - x)w = \frac{1 - x}{1 - 2x^2} (2K) > 2K.$$ 

This means that the expected wage payment is higher than in the case of two agents.

**Conclusion**

- If we have two agents, we can implement the first best with expected wage payment of $2K$, no matter if we are facing the limited liability or not.
- If we have one agent and $x \geq 1/2$, the first best cannot be implemented.
- If we have one agent and $x < 1/2$, we can implement the first best. The expected wage payment is $2K$ if we don’t have limited liability constraint and is greater than $2K$ if we have limited liability constraint.
In the case of two agents, the result for part (c) is unchanged because under the optimal contract given in (c), agents don’t have incentives to hide the information.

However, in the case of one agent, under the optimal contract given in (c), the agent wants to show only one information and the mechanism doesn’t work.