Solution to PS 5

Problem 1: Adaptation

(a) The first best decision rule is to choose \( d=0 \) in \( s_1 \) and \( d=1 \) in \( s_2 \).

(b) If we allocate the decision right to A, the per period expected total payoff
\[
E[U_A + U_B] = \frac{1}{2}(1 + 4) = \frac{5}{2}
\]
If we allocate the decision right to B, the per period expected total payoff is
\[
E[U_A + U_B] = \frac{1}{2}(6 + 2) = 4
\]
Therefore, the second best allocation of the decision right goes to B.

(c) (i) Note that A owns the decision right at time 0 of period 1. We assume that whoever has the decision right in a period inherits the decision right at the beginning of the next period. Consider the following trigger strategy:

On the equilibrium path:
In each period, (1) A obtains the decision right through bargaining.
(2) B pays A \( k>0 \)
(3) A chooses \( d=0 \) in \( s_1 \) and \( d=1 \) in \( s_2 \).
(4) B pays A \( m(s) \)

Off the equilibrium path:
(1) No bargaining occurs. The owner of the decision right keeps the decision right.
(2) No transfers of money take place between parties.
(3) The owner of the decision right chooses a decision favorable to him.
(4) No transfers take place.

(ii) Denote \( U^F_A, U^F_B \) as the equilibrium payoff for A and B. \( U^F_A + U^F_B = U^{FB} \)
Define \( U^N_A, U^N_B \) as the off-equilibrium payoff for A and B. \( U^N_A + U^N_B = U^{NE} \)
For the strategy above to be a SPE, we need to check the following IC constraints.
First, B is willing to give \( k \). This is the same as
\[
U^N_B - U^F_B \leq 0
\]
Second, B is willing to give \( m(s) \) to A for all \( s \), i.e.
\[
m(s) \leq \frac{1}{r} (U^F_B - U^N_B)
\]
The left hand side is B’s current period gain if he fails to pay for \( m(s) \). The right hand is the loss of continuation payoff.
Third, A will want to choose \( d=0 \) in \( s_1 \), i.e.
\[
1 - m(s_1) \leq \frac{1}{r} (U^F_A - U^N_A)
\]
The left hand side is A’s gain in the current period if he deviates. A receives 1 by choosing \(d=1\) and loses the payment from B \(m(s_i)\). The right hand side is the loss of continuation value.

By adding the above inequalities, a necessary condition is
\[
1 \leq \frac{1}{r} (U^{FB} - U^{NE}) = \frac{1}{r} (5 - \frac{5}{2})
\]

Note that in this equilibrium A has the decision right on the off-equilibrium path, so
\[
U_A^{NE} = \frac{5}{2}, U_B^{NE} = 0.
\]

Note that the necessary condition is also sufficient. In particular, consider the case that B always pays \(m\) to A after the bargaining. And no other transfer is involved. For B to be willing to pay for \(m\), we need,
\[
-m + \frac{1}{2} (6) + \frac{1}{r} (-m + \frac{1}{2} (6)) \geq 0
\]
where the left hand side is \(U_B^{FB}\) and the right hand side is \(U_B^{NE}\). In other words, we need \(m \leq 3\)

On the other hand, A’s incentive constraint is
\[
1 + \frac{1}{r} (-\frac{1}{2} 5) \leq \frac{1}{r} (m + \frac{1}{2})
\]

By choosing \(d=0\), A receives 0 from the action and a continuation payoff of \(\frac{1}{r} (m + \frac{1}{2} \times 4)\).

If A chooses \(d=1\), he receives 1 from the action, and her continuation payoff will be \(\frac{1}{r} 5\).

The inequality above can be rewritten as
\[
\frac{1}{r} (m - \frac{1}{2}) \geq 1.
\]

Let \(m=3\), we see that the maximum value of \(r\) is \(\frac{5}{2}\).

(iii) If the initial assignment of decision right goes to B, then \(U_A^{NE} = 0, U_B^{NE} = 4\). A similar necessary condition as in (ii) says that
\[
\frac{1}{r} (U^{FB} - U^{NE}) = \frac{1}{r} (5 - 4)
\]
where \(2\) is B’s gain from deviation in \(s_2\). Therefore, \(r \leq \frac{1}{2}\). As a result, it is better to allocate the initial decision right to A.
Problem 2: Contracting for Control

(a) Let \( s^* \) be the cutoff state in which A and B have identical payoff:
\[
\sigma_B s^* + \rho_B = \sigma_A s^* + \rho_A
\]
then the first best rule is to pick
\[
d = \begin{cases} 
  d_A & \text{if } s < s^* \\
  d_B & \text{if } s \geq s^* 
\end{cases}
\]
The expected total payoff is then
\[
(s^* - s_L)[\rho_A + \frac{1}{2} \sigma_A (s_L + s^*)] + (s_H - s^*)[\rho_B + \frac{1}{2} \sigma_B (s_H + s^*)]
\]
where \( s^* = \frac{\rho_A - \rho_B}{\sigma_B - \sigma_A} \).

(b) In the spot version, if A controls the decision right, her expected surplus is
\[
\rho_A + \frac{1}{2} \sigma_A (s_L + s_H)
\]
If B controls the decision right, her expected payoff is
\[
\rho_B + \frac{1}{2} \sigma_B (s_L + s_H)
\]
Therefore, A controls the decision right if and only if
\[
\rho_A - \rho_B \geq \frac{1}{2} (\sigma_B - \sigma_A) (s_L + s_H)
\]

(c) Consider the following trigger strategy.
On the equilibrium path:
- In each period, (1) A controls the decision right.
- (2) A pays B t.
- (3) A chooses the first best decision.
- (4) A pays B T(d,s)
Off the equilibrium path:
- (1) The owner of the decision right keeps the decision right.
- (2) No transfers of money take place between parties.
- (3) The owner of the decision right chooses a decision favorable to her.
- (4) No transfers take place.
For the trigger strategy to be a SPE, there are several possibilities that A will deviate.
- (i) Fail to pay t.
- (ii) Fail to carry out the correct decision.
- (iii) Fail to pay T(d,s)
Denote \( U^F_B \), \( U^F_A \) as the equilibrium payoff for A and B. \( U^F_B + U^F_A = U^F \) Define \( U^A_N, U^B_N \) as the off-equilibrium payoff for A and B. \( U^A_N + U^B_N = U^N \).
Then the IC for (i) is like an IR constraint, i.e.
\[
U^F_A \geq U^N_A
\]
Denote $\Delta_A(s)$ as the extra gain for A from deviation from the first best decision. Then the IC for (ii) is

$$\Delta_A(s) + T(d, s) \leq \frac{1}{r}(U^{FB}_A - U^{NE}_A)$$

Note that IC in (ii) implies the IC in (iii).\( (T(d, s) \leq \frac{1}{r}(U^{FB}_A - U^{NE}_A) )\)

Now for B, there are two places he might want to deviate. First, he might fail to pay $-t$. This is like an IR constraint that requires

$$U^{FB}_B \geq U^{NE}_B$$

Second, he may deviate is failure to pay $-T(d, s)$. The IC for B therefore is

$$-T(d, s) \leq \frac{1}{r}(U^{FB}_B - U^{NE}_B)$$

Adding the two inequalities above shows that a necessary condition for the trigger strategy to be a SPE is

$$\sup_s \Delta_A(s) \leq \frac{1}{r}(U^{FB}_A - U^{NE}_A)$$

As in problem 1, it is easy to see that this is a sufficient condition as well. To see this, let $T(d, s) = 0$ for all $d, s$, and let $t = -(s_H - s^*)(\rho_B + \frac{1}{2}\sigma_B(s_H + s^*))$, so $U^{FB}_B = U^{NE}_B = 0$. It is straightforward to check that A will not deviate in this situation provided

$$\sup_s \Delta_A(s) = \Delta_A(s^*) = \frac{\sigma_B\rho_B - \sigma_A\rho_A}{\sigma_B - \sigma_A} \leq \frac{1}{r}(U^{FB} - U^{NE})$$

(d) From the analysis in (c), we see that the maximum discount rate for first best when decision right goes to A is

$$r_A = \frac{U^{FB}_A - U^{NE}_A}{\sup_s \Delta_A(s)}$$

When B has the decision right, the maximum discount rate for first best is

$$r_B = \frac{U^{FB}_B - U^{NE}_B}{\sup_s \Delta_B(s)}$$

A should be given the decision right if and only if $r_A \geq r_B$. In this case, since $\sup_s \Delta_A(s) = \sup_s \Delta_B(s)$, the maximum temptation to deviate happens at $s^*$, A should have the decision right if and only if $U^{NE}_A \leq U^{NE}_B$, i.e.

$$\rho_A - \rho_B \leq \frac{1}{2}(\sigma_B - \sigma_A)(s_L + s_H)$$

(e) In this case, the same analysis goes through. We have

$$r_A = \frac{U^{FB}_A - U^{NE}_A}{\sup_s \Delta_A(s)} \quad \text{and} \quad r_B = \frac{U^{FB}_B - U^{NE}_B}{\sup_s \Delta_B(s)}$$

And A should be given the decision right if and only if $r_A \geq r_B$. The only difference is that $\sup_s \Delta_B(s)$ now occurs in $s_L$ instead of $s^*$. 
(a) To characterize the SPEs of this game, we start backwards. In stage (3), the only action profile that forms a Nash equilibrium of the subgame is

\[ m_A = m_B = 0 \]

Therefore in stage (2), we must have

\[ d_A^*(s) \in \arg \max_{d_a} \ U_A(d_A, d_B^*(s), s) \]

and

\[ d_B^*(s) \in \arg \max_{d_b} \ U_B(d_B, d_B^*(s), s) \]

(b) (i): The first best decision rule \( d_{FB}^* \) satisfies that

\[ (d_A^{FB}(s), d_B^{FB}(s)) \in \arg \max_{d_A, d_B} \ (U_A(d_A^{FB}(s), d_B^{FB}(s), s) + U_B(d_A^{FB}(s), d_B^{FB}(s), s)) \]

(ii): Consider the following trigger strategy:

On the equilibrium path:
1. Players choose \( d_{FB}^* \)
2. Player A pays B \( m(s) \) respectively.

Off the equilibrium path:
1. Players choose \( d_{NE}^* \)
2. Players pay \( m_A - m_B = 0 \).

Note that in describing the strategy, we can always assume \( m_B = 0 \).

Now define the maximum gain from deviation for player \( i \) in state \( s \) as

\[ \Delta_i(s) = \max_{d_i} \ U_i(d_i(s), d_j^{FB}(s), s) + U_i(d_A^{FB}(s), d_B^{FB}(s), s) \]

And let \( U_A^{FB}, U_B^{FB} \) as the equilibrium payoff for A and B. \( U_A^{FB} + U_B^{FB} = U^{FB} \) Define \( U_A^{NE}, U_B^{NE} \) as the off-equilibrium payoff for A and B. \( U_A^{NE} + U_B^{NE} = U^{NE} \)

Then as in problem 1 and 2, the necessary conditions become

\[ \Delta_A(s) + m(s) \leq \frac{1}{r} (U_A^{FB} - U_A^{NE}) \]

\[ \Delta_B(s) - m(s) \leq \frac{1}{r} (U_B^{FB} - U_B^{NE}) \]

Then a necessary condition is

\[ \sup_s (\Delta_A(s) + \Delta_B(s)) \leq \frac{1}{r} (U^{FB} - U^{NE}) \]

In general, this is a sufficient condition as well. To see this, denote

\[ r = \frac{U^{FB} - U^{NE}}{\sup_s (\Delta_A(s) + \Delta_B(s))} \]

Now we can choose \( m(s) \), for \( s = 1, 2, \ldots, n \) such that

\[ r(\Delta_A(s) + m(s)) = (U_A^{FB} - U_A^{NE}) \]

Note that \( U_A^{FB} \) is a function of \( m(s) \) as well.
This can be done in general, because we have $n$ unknowns for $n$ equations.

With these $m(s)$ chosen, we see that

$$\Delta_s + m(s) = \frac{1}{r}(U^F_B - U^NE_B) \leq \frac{1}{r}(U^F_A - U^NE_A)$$

for all $s$.

Therefore, the ICs for A are all satisfied.

We now claim that with these $m(s)$ chosen, we must also have

$$r(\Delta_A(s) + m(s)) \leq (U^F_B - U^NE_B)$$

This would imply that the ICs for B is satisfied as well and thus prove the sufficiency.

We prove the claim above by contradiction. If the claim is false, then there exists one state $s'$ such that

$$r(\Delta_A(s') + m(s')) > (U^F_B - U^NE_B)$$

Since

$$r(\Delta_A(s') + m(s')) = (U^F_A - U^NE_A)$$

Summing the two equations above gives that

$$r(\Delta_A(s') + \Delta_B(s')) > (U^F_B - U^NE_B)$$

In other words,

$$r > \frac{(U^F_B - U^NE_B)}{\Delta_A(s') + \Delta_B(s')} \geq \frac{U^F_B - U^NE_B}{\sup(\Delta_A(s) + \Delta_B(s))}$$

And this contradicts the definition of $r$. So we prove the claim and thus prove the sufficiency.

Note that this way of proving the sufficiency is not very constructive because it relies on the existence of solutions for $n$ linear equations.

(c): Trigger Strategy:

On the equilibrium path:

Players choose $d^F_B(s)$

Off the equilibrium path:

Players choose $d^NE(s)$

To determine the larger $r$, we see that for player A, the ICs are for all $s$,

$$\Delta_A(s) \leq \frac{1}{r}(U^F_A - U^NE_A)$$

For player B, the ICs are for all $s$,

$$\Delta_B(s) \leq \frac{1}{r}(U^F_B - U^NE_B)$$

Therefore, the maximum discount

$$r = \min \left( \frac{U^F_i - U^NE_i}{\sup(\Delta_i(s))} \right)$$