Due September 23 in Section (2pm)

1. Inspired by "The Two Towers."
   A signal is transmitted from city A to city B via a series of sentries on a chain of $N$ mountaintops. If the sentry on mountaintop 2 sees the signal fire from mountaintop 1, he will light his own fire for the sentry on mountaintop 3 to see, etc.
   There are $N$ sentries to post to these mountaintops, but $K$ are drunkards and will always fail to perform their duty. The remaining $N - K$ will always light their signal fires at the right time. You cannot distinguish between drunk and sober sentries, so the sentries are posted to mountaintops at random. Fortunately, the sentry on mountaintop 3 can also see mountaintop 1 directly, so if the sentry on mountaintop 2 is a drunkard, but the sentries on mountaintops 1 and 3 are both sober, the signal will get through, at least as far as mountaintop 3. Similarly, the sentry on mountaintop 4 can see mountaintops 2 and 3, but not mountaintop 1. And so on (5 can see 3 and 4, but not 1 or 2, etc.)
   Therefore, as long as no two consecutive sentries are drunkards, the signal will get through. (The sentries in cities A and B are always on duty, and the sentries in city B can see both mountaintop $N$ and mountaintop $N - 1$.)
   What is the probability that the signal will get through?

2. My telephone rings 12 times each week, the calls being randomly distributed among the 7 days. What is the probability that I get at least one call each day?

3. Three prisoners A, B, and C know that exactly two of them are going to be executed, but they do not know which two. Prisoner A knows that the jailer will not tell him whether or not he is going to be executed. He therefore asks the jailer to tell him the name of one prisoner other than A himself who will be executed. The jailer responds that B will be executed. Upon receiving this response, prisoner A reasons as follows: before he spoke to the jailer, the probability was $\frac{2}{3}$ that he will be executed. After speaking to the jailer, he knows that either he or C will be executed. Hence, the probability that he will be executed is now only $\frac{1}{2}$. Hence, merely by asking the jailer his question, the prisoner reduced the probability that he would be executed from $\frac{2}{3}$ to $\frac{1}{2}$, because he could go through exactly the same reasoning regardless of which answer the jailer gave. Discuss what is wrong with prisoner A's reasoning. (Do not write more than 10 lines.)

4. Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define
   
   $p_0 = P(0 \text{ heads occur})$,
   
   $p_1 = P(1 \text{ head occurs})$,
   
   $p_2 = P(2 \text{ heads occur})$.

   Can $u$ and $w$ be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

5. Prove that if $P(A) > 0$ and $P(B) > 0$, then:
   (a) If $A$ and $B$ are mutually exclusive, they cannot be independent.
   (b) If $A$ and $B$ are independent, they cannot be mutually exclusive.