Final Exam-Fall 2001

Instructions: This is a 180-minute open-book, open-notes exam.

1. Assume that \( \varepsilon_t \) and \( \eta_t \) are sequences of iid \( N(0,1) \) random variables with \( E\varepsilon_t\eta_s = 0 \) for all \( t \) and \( s \). Let

\[
x_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}
\]

where \( \sum |c_j| j < \infty \) and

\[
u_t = \sum_{j=0}^{\infty} \gamma_j \eta_{t-j}
\]

with \( \sum |\gamma_j| j < \infty \). Assume that \( y_t \) is generated by

\[
y_t = \beta x_t + u_t
\]

and that we observe a sample of size \( n \) of observations \( \{y_t, x_t\}_{t=1}^{n} \).

a) Consider the OLS estimator

\[
\hat{\beta} = \frac{\sum_{t=1}^{n} x_t y_t}{\sum_{t=1}^{n} x_t^2}
\]

Is \( \hat{\beta} \) consistent? Find a limiting distribution of an appropriately centered and scaled version of \( \hat{\beta} \). What is the variance of the limiting distribution of \( \hat{\beta} \)?

b) Consider the random variable \( z_t = x_t u_t \). Find the autocovariance function of \( z_t \) as a function of the parameters \( c_j \) and \( \gamma_j \). Find the spectral density of \( z_t \).

c) Express the asymptotic variance of \( \hat{\beta} \) found in a) in terms of the spectral densities of \( z_t \) and \( x_t \) (Hint: use the fact that \( f_x(\lambda) = (2\pi)^{-1} \left| C(e^{-i\lambda}) \right|^2 \) where \( C(L) = \sum_{j=0}^{\infty} c_j e^{-i\lambda j} \)).

d) How does your result in c) simplify if \( \gamma_j = 0 \) for \( j \neq 0 \)?

e) Derive a test of the hypothesis that \( u_t \) is an iid sequence against the alternative that \( \gamma_j \neq 0 \) for at least one \( j \leq q \) where \( q \) is a fixed and known positive integer?

2. Let \( x_t \) be a weakly stationary process with \( E x_t = 0 \). Consider the filtered process \( y_t \) where

\[
y_t = \frac{1}{3} (x_t + x_{t-1} + x_{t-2})
\]

a) Find the spectral density of \( y_t \) in terms of the spectral density \( f_x(\lambda) \) of \( x_t \) and the power transfer function of the filter.

b) Describe the properties of the filter by finding the gain function and the phase shift of the filter.

c) Now consider the extracted 'cyclical' component \( y_t^c \) of \( x_t \) defined as

\[
y_t^c = x_t - y_t.
\]

Find the filter that transforms \( x_t \) into \( y_t^c \). Assume that \( x_t \) is a unit root process of the form \( x_t = x_{t-1} + u_t \) where \( u_t \) is iid \( N(0,1) \) and \( x_0 = 0 \). Is \( y_t^c \) stationary? Prove your answer.
d) Consider the filtered series $y^c_t$. Discuss the optimality properties (or lack thereof) of $y^c_t$ if the goal was to extract the components of $x_t$ that generate the spectrum in a band $[a,b]$. How does your answer depend on $a$ and $b$. (Hint: you can answer this part informally. You may find it useful to compare $y^c_t$ to the approximately optimal filter according to Christiano and Fitzgerald when $x_t$ is generated by

$$x_t = x_{t-1} + u_t - u_{t-1}.$$ 

Only spend time on this last point if you have time at the end of the exam.)

3. Assume that $x_t = \alpha x_{t-1} + \varepsilon_t - \theta \varepsilon_{t-1}$ where $\varepsilon_t \sim \text{iid} \ N(0,1)$, $|\theta| < 1$ and $x_0 = 0$. Consider a regression of $x_t$ onto $x_{t-1}$, i.e. an estimator

$$\hat{\alpha} = \frac{\sum_{t=2}^{n} x_t x_{t-1}}{\sum_{t=2}^{n} x^2_{t-1}}.$$ 

a) Assume that $|\alpha| < 1$. Is $\hat{\alpha}$ consistent for the parameter value $\alpha$? Prove your answer. Find a limiting distribution for $\hat{\alpha}$. Is $\hat{\alpha}$ asymptotically efficient?

b) Now assume that $\alpha = 1$. Is $\hat{\alpha}$ consistent in this case. Find the limiting distribution of $\hat{\alpha}$.

c) How could you use your distributional result in b) to approximate the finite sample bias of $\hat{\alpha}$. (Hint: no exact calculation is expected here. A rough outline of the procedure is sufficient).

d) Propose a test of $H_1: |\alpha| < 1$ against the null hypothesis $H_0: \alpha = 1$? Derive the limiting distribution of your test statistic under the null. How can you make the distribution of your test statistic nuisance parameter free?