1. An Optimal Taylor Rule

Consider an economy with Calvo-type staggered price setting whose equilibrium dynamics are described by the system:

\[ y_t = E_t\{\bar{y}_{t+1}\} - \frac{1}{\sigma} (r_t - E_t\{\pi_{t+1}\} - \rho) + \varepsilon_t \]

\[ \pi_t = \beta E_t\{\pi_{t+1}\} + \kappa \bar{y}_t + u_t \]

where \{\varepsilon_t\} and \{u_t\} represent sequences of i.i.d., mutually uncorrelated, demand and supply disturbances, with variances given by \( \text{var}(\varepsilon) \) and \( \text{var}(u) \).

Assume that the monetary authority decides to adopt a simple Taylor rule of the form

\[ r_t = \rho + \phi \pi_t \]

a) Solve for the equilibrium processes for the output gap and inflation, as a function of the exogenous supply and demand shocks.

b) Determine the value of the inflation coefficient \( \phi \) which minimizes the central bank’s loss function:

\[ \alpha \text{var}(\bar{y}_t) + \text{var}(\pi_t) \]

c) Discuss and provide intuition for the dependence of the optimal inflation coefficient on \( \alpha \) and the variance ratio \( \frac{\text{var}(\varepsilon)}{\text{var}(u)} \). What assumptions on parameter values would warrant an extremely aggressive response to inflation \( (\phi \rightarrow +\infty) \)? Explain.

2. Monetary Policy, Optimal Steady State Inflation and the Zero Lower Bound (40 points)

Consider the equilibrium conditions of a standard new Keynesian model:

\[ y_t = E_t\{\bar{y}_{t+1}\} - \frac{1}{\sigma} (r_t - E_t\{\pi_{t+1}\} - \rho) + \varepsilon_t \]

and

\[ \pi_t - \pi = \beta E_t\{(\pi_{t+1} - \pi)\} + \kappa \bar{y}_t + u_t \]

where \( \bar{y}_t \) is the output gap, \( \pi_t \) denotes inflation, \( r_t \) is the nominal rate, and \( \pi \) is steady state inflation. The disturbances \( \varepsilon_t \) and \( u_t \) represent demand and cost-push shocks, and are assumed to follow independent and serially uncorrelated normal distributions with zero mean and variances \( \sigma^2_\varepsilon \) and \( \sigma^2_u \) respectively.

Assume that the loss function for the monetary authority is given by
\[ \theta \pi + E_0 \sum_{t=0}^{\infty} \beta^t \left[ \alpha \tilde{y}_t^2 + (\pi_t - \pi)^2 \right] \]

where the first term is assumed to capture the costs of steady state inflation.

(a) Derive the optimal policy under discretion (i.e., the time-consistent policy, resulting from period-by-period maximization) —including the choice of steady state inflation \( \pi^- \), subject to the constraint that the interest rate hits the zero-bound constraint with only a 5 percent probability.

(b) Derive an interest rate rule that would implement the optimal allocation derived in (a) as the unique equilibrium.