Topic #13

16.31 Feedback Control

State-Space Systems

- Full-state Feedback Control
- How do we change the poles of the state-space system?
- Or, even if we can change the pole locations.
- Where do we change the pole locations to?
- How well does this approach work?
**Full-state Feedback Controller**

- Assume that the single-input system dynamics are given by
  \[
  \dot{x} = Ax + Bu \\
  y = Cx
  \]
  so that \( D = 0 \).

  - The multi-actuator case is quite a bit more complicated as we would have many extra degrees of freedom.

- Recall that the system poles are given by the eigenvalues of \( A \).
  - Want to use the input \( u(t) \) to modify the eigenvalues of \( A \) to change the system dynamics.

- Assume a full-state feedback of the form:
  \[
  u = r - Kx
  \]
  where \( r \) is some reference input and the gain \( K \) is \( \mathcal{R}^{1 \times n} \).
  - If \( r = 0 \), we call this controller a regulator.

- Find the closed-loop dynamics:
  \[
  \dot{x} = Ax + B(r - Kx) \\
  = (A - BK)x + Br \\
  = A_{cl}x + Br \\
  y = Cx
  \]
• **Objective:** Pick $K$ so that $A_{cl}$ has the desired properties, *e.g.*, 
  - $A$ unstable, want $A_{cl}$ stable
  - Put 2 poles at $-2 \pm 2j$

• Note that there are $n$ parameters in $K$ and $n$ eigenvalues in $A$, so it looks promising, but what can we achieve?

• **Example #1:** Consider:

  \[
  \dot{x} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
  \]

  - Then
  \[
  \det(sI - A) = (s - 1)(s - 2) - 1 = s^2 - 3s + 1 = 0
  \]
  so the system is unstable.

  - Define $u = - \begin{bmatrix} k_1 & k_2 \end{bmatrix} x = -Kx$, then
  \[
  A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 1 & 2 \end{bmatrix}
  \]

  - So then we have that
  \[
  \det(sI - A_{cl}) = s^2 + (k_1 - 3)s + (1 - 2k_1 + k_2) = 0
  \]

  - Thus, by choosing $k_1$ and $k_2$, we can put $\lambda_i(A_{cl})$ anywhere in the complex plane (assuming complex conjugate pairs of poles).
• To put the poles at $s = -5, -6$, compare the desired characteristic equation
\[(s + 5)(s + 6) = s^2 + 11s + 30 = 0\]
with the closed-loop one
\[s^2 + (k_1 - 3)x + (1 - 2k_1 + k_2) = 0\]
to conclude that
\[
\begin{cases}
  k_1 - 3 = 11 \\
  1 - 2k_1 + k_2 = 30
\end{cases}
\]
so that $k_1 = 14$ and $k_2 = 57$

so that $K = \begin{bmatrix} 14 & 57 \end{bmatrix}$, which is called Pole Placement.

• Of course, it is not always this easy, as the issue of controllability must be addressed.

• Example #2: Consider this system:
\[
\dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]
with the same control approach
\[
A_{cl} = A - BK = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 - k_1 & 1 - k_2 \\ 0 & 2 \end{bmatrix}
\]
so that
\[
\det(sI - A_{cl}) = (s - 1 + k_1)(s - 2) = 0
\]
So the feedback control can modify the pole at $s = 1$, but it cannot move the pole at $s = 2$.

• The system cannot be stabilized with full-state feedback control.
• What is the reason for this problem?
  – It is associated with loss of controllability of the $e^{2t}$ mode.

• Consider the basic controllability test:
  \[
  \mathcal{M}_c = \begin{bmatrix}
  B & AB
  \end{bmatrix} = \begin{bmatrix}
  1 & 1 \\
  0 & 2
  \end{bmatrix} \begin{bmatrix}
  1 \\
  0
  \end{bmatrix}
  \]
  So that $\text{rank } \mathcal{M}_c = 1 < 2$.

• Consider the modal test to develop a little more insight:
  \[
  A = \begin{bmatrix}
  1 & 1 \\
  0 & 2
  \end{bmatrix}, \text{ decompose as } AV = V\Lambda \Rightarrow \Lambda = V^{-1}AV
  \]
  where
  \[
  \Lambda = \begin{bmatrix}
  1 & 0 \\
  0 & 2
  \end{bmatrix}, \quad V = \begin{bmatrix}
  1 & 1 \\
  0 & 1
  \end{bmatrix}, \quad V^{-1} = \begin{bmatrix}
  1 & -1 \\
  0 & 1
  \end{bmatrix}
  \]
  Convert
  \[
  \dot{x} = Ax + Bu \quad \Rightarrow \quad \dot{z} = \Lambda z + V^{-1}Bu
  \]
  where $z = [z_1 \ z_2]^T$. But:
  \[
  V^{-1}B = \begin{bmatrix}
  1 & -1 \\
  0 & 1
  \end{bmatrix} \begin{bmatrix}
  1 \\
  0
  \end{bmatrix} = \begin{bmatrix}
  1 \\
  0
  \end{bmatrix}
  \]
  so that the dynamics in modal form are:
  \[
  \dot{z} = \begin{bmatrix}
  1 & 0 \\
  0 & 2
  \end{bmatrix} z + \begin{bmatrix}
  1 \\
  0
  \end{bmatrix} u
  \]

• With this zero in the modal $B$-matrix, can easily see that the mode associated with the $z_2$ state is **uncontrollable**.
  – **Must assume that the pair** $(A, B)$ **are controllable.**
Ackermann’s Formula

- The previous outlined a design procedure and showed how to do it by hand for second-order systems.
  - Extends to higher order (controllable) systems, but tedious.

- **Ackermann’s Formula** gives us a method of doing this entire design process is one easy step.
  \[ K = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} \mathcal{M}_c^{-1} \Phi_d(A) \]
  where
  - \( \mathcal{M}_c = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix} \)
  - \( \Phi_d(s) \) is the characteristic equation for the closed-loop poles, which we then evaluate for \( s = A \).
  - It is explicit that the system must be controllable because we are inverting the controllability matrix.

- Revisit example # 1: \( \Phi_d(s) = s^2 + 11s + 30 \)
  \[ \mathcal{M}_c = \begin{bmatrix} B \mid AB \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix} \]
  So
  \[ K = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \end{bmatrix}^{-1} \left( \begin{bmatrix} 1 & 1 \end{bmatrix}^2 + 11 \begin{bmatrix} 1 & 1 \end{bmatrix} + 30I \right) \]
  \[ = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 43 & 14 \n 14 & 57 \end{bmatrix} = \begin{bmatrix} 14 & 57 \end{bmatrix} \]

- Automated in Matlab: `place.m` & `acker.m` (see `polyvalm.m` too)
• Where did this formula come from?

• For simplicity, consider a third-order system (case #2), but this extends to any order.

\[
A = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}
\]

- See key benefit of using control canonical state-space model
- This form is useful because the characteristic equation for the system is obvious \( \Rightarrow \det(sI - A) = s^3 + a_1 s^2 + a_2 s + a_3 = 0 \)

• Can show that

\[
A_{cl} = A - BK = \begin{bmatrix} -a_1 & -a_2 & -a_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}
= \begin{bmatrix} -a_1 - k_1 & -a_2 - k_2 & -a_3 - k_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}
\]

so that the characteristic equation for the system is still obvious:

\[
\Phi_{cl}(s) = \det(sI - A_{cl}) = s^3 + (a_1 + k_1)s^2 + (a_2 + k_2)s + (a_3 + k_3) = 0
\]
• We then compare this with the desired characteristic equation developed from the desired closed-loop pole locations:

\[ \Phi_d(s) = s^3 + (\alpha_1)s^2 + (\alpha_2)s + (\alpha_3) = 0 \]

to get that

\[
\begin{align*}
    a_1 + k_1 &= \alpha_1 \\
    \vdots & \vdots \\
    a_n + k_n &= \alpha_n
\end{align*}
\]

\[
\begin{align*}
    k_1 &= \alpha_1 - a_1 \\
    \vdots & \vdots \\
    k_n &= \alpha_n - a_n
\end{align*}
\]

• To get the specifics of the Ackermann formula, we then:

  – Take an arbitrary \( A, B \) and transform it to the control canonical form \( (x \sim z = T^{-1}x) \)
  – Solve for the gains \( \hat{K} \) using the formulas above for the state \( z \) \( (u = \hat{K}z) \)
  – Then switch back to gains needed for the state \( x \), so that

\[
K = \hat{K}T^{-1}
\]

\[
(u = \hat{K}z = Kx)
\]

• Pole placement is a very powerful tool and we will be using it for most of this course.