Solution by Dynamic Programming

Principle of optimality:

Any portion of an optimal trajectory is an optimal trajectory.

Optimal "cost-to-go" is

\[
J^*(x(t), t) = \min_u \int_t^T \left[ x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] dt
\]

\[
= \min_u \left( \left[ x(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] + J^* (x(t+dt), t+dt) \right)
\]

where

\[
x(t+dt) = x(t) + \left[ A(t) x(t) + B(t) u(t) \right] dt
\]
If we knew $J^*(x, t+dt)$ for all $x$, can choose best $u$ at time $t$.

Let's guess that $J^*(x, t) = x^T P(t) x$ and need to find!

(Because $J$ is quadratic)

Then

$$x^T(t) P(t) x(t) =$$

$$\min_u \left\{ \left[ x^T(t) Q(t) x(t) + u^T(t) R(t) u(t) \right] dt + \frac{1}{2} \left( x(t) + \left[ A(t) x(t) + B(t) u(t) \right] dt \right)^T P(t + dt) \left( x(t) + \left[ A(t) x(t) + B(t) u(t) \right] dt \right) \right\}$$

Keep only $O(1)$ and $O(dt)$ terms:

$$x^T P x =$$

$$\min_u \left\{ \left[ x^T Q x + u^T R u \right] dt + x^T P x + x^T P x dt \right\}$$

$$+ x^T P (A x + B u) dt + (A x + B u)^T P x dt$$

To minimize,

$$\frac{d}{du} \{ \cdot \} = 0 = \left( 2 R u + 2 B^T P x \right) dt$$
\[ u = -R^{-1}B^TPx \]

is the optimum control (if form of \( J \) is correct)

\[ u(t) = -R^{-1}(t)B^T(t)P(t)x(t) = -F(t)x(t) \]

\[ F(t) = R^{-1}(t)B^T(t)P(t) \]

So,

\[ \dot{x}^TPx = (x^TQx + x^TPBR^{-1}R^{-1}B^TPx)dt + x^TPx \]
\[ + x^TPdx + x^TP(Ax - BR^{-1}B^TPx)dt \]
\[ + (Ax - BR^{-1}B^TPx)^TPxdt \]

\[ 0 = x^T(G + PBR^{-1}B^TP + \dot{P} + PA - PBR^{-1}B^TP \]
\[ + ATP - PBR^{-1}B^TP)x \]

Therefore, \( P(t) \) satisfies

\[ -\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + Q(t) - P(t)B(t)R^{-1}B(t)P(t) \]

"Riccati Equation"

Integrate backwards in time. Final condition:

\[ P(T) = 0 \]
The steady-state solution

In many cases:

- \( T = \infty \) (or \( T \) large)
- \( A, B, Q, R \) constant

In this case, expect \( P(t) \to \text{const. as} \quad T-t \to \infty \).

If \( P(t) \) reaches a steady state, \( P \) satisfies

the "algebraic Riccati equation" (ARE)

\[
0 = A^TP + PA + Q - PBR^{-1}B^TP
\]

(like a quadratic eq'n)

\( \Rightarrow \) more than 1 solution

and the optimal gain is

\( F = R^{-1}B^TP \)

Theorem: If \( (A,B) \) is stabilizable and \( (A, Q^{1/2}) \)

is detectable,

\[
\lim_{T-t \to \infty} P(t) = \overline{P} \quad \text{exists},
\]

which is the unique solution of the ARE

for which \( F \succ 0 \).
The Steady-State Solution

In many cases:

- $T = \infty$ (or $T$ large)
- $A, B, Q, R$ constant

In this case, expect $P(t) \to \text{const.}$ as $T-t \to \infty$.

If $P(t)$ reaches a steady state, $P$ satisfies the "algebraic Riccati equation" (ARE)

$$0 = A^T P + P A + Q - P B R^{-1} B^T P$$

Like a quadratic eq'n, $\Rightarrow$ more than 1 solution

and the optimal gain is

$$F = R^{-1} B^T P$$

Theorem: If $(A, B)$ is stabilizable and $(A, Q^{1/2})$ is detectable,

$$\lim_{T-t \to \infty} P(t) = \bar{P} \text{ exists,}$$

which is the unique solution of the ARE for which $P \succ 0$. 

Note: Didn't go well.

See Lecture 23.
Solving the Riccati Equation

Riccati equation is intimately related to Hamiltonian system

\[
\begin{pmatrix}
\dot{x} \\
\dot{p}
\end{pmatrix} =
\begin{pmatrix}
A & -BR^{-1}B^T \\
-Q & -A^T
\end{pmatrix}
\begin{pmatrix}
x \\
p
\end{pmatrix}
\]

Why? Hamiltonian matrix, $H$.

1) Could solve optimal control problem by calculus of variations. $p$ is the Lagrange multiplier, and can show that $\dot{p} = P \dot{x}$

2) Define $p = Px$, and see what happens.

\[
\dot{x} = Ax + Bu = Ax - BFx
\]

\[
= Ax - BR^{-1}B^TPx
\]

\[
\dot{x} = Ax - BR^{-1}B^TPp
\]

\[
\dot{p} = \frac{d}{dt}(Px) = \dot{p}x + P \dot{x}
\]

\[
= -(A^TP + PA + Q - PBR^{-1}B^TP)x + P(Ax + Bx)
\]

\[
\dot{p} = -Qx - A^TPx = -Qx - A^TPp
\]
\[
\begin{pmatrix}
\dot{x} \\
\phi
\end{pmatrix} =
\begin{pmatrix}
A & -BR^{-1}B^T \\
-Q & -A^T
\end{pmatrix}
\begin{pmatrix}
\chi \\
P
\end{pmatrix}
\]

Riccati equation \implies Hamiltonian.

How do we go the other way?

We can show that \( \Phi_H(s) = \text{det}(sI - H) = \Phi_H(-s) \)

\( \implies \) if \( \lambda \) is an eigenvalue of \( H \), so is \( -\lambda \).

**Proof** Take \( T = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \). Then

\[
TH^{-1}T^{-1} = \begin{bmatrix}
-A^T & Q \\
+BR^{-1}B^T & A
\end{bmatrix} = -H^T
\]

\( \implies \lambda(H) = \lambda(TH^{-1}T^{-1}) = \lambda(-H^T) = -\lambda(H) = -\lambda(H) \)

So half the poles of \( H \) are stable. Those must correspond to stable regulator poles.

So,

1) Find the eigenvalues, eigenvectors of \( H \).
2) Keep only the stable ones.
3) Let \( \Delta = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \)