The Linear Quadratic Gaussian (LQG) Problem

We are now ready to solve the real problem:

Find the compensator for the system

\[ \dot{x} = Ax + Bu + Gw \]
\[ y = Cx + v \]
\[ z = Ex \]

where \( w(t), v(t) \) are white, that minimizes the cost

\[ J = E \left[ \int_{t_1}^{t_2} (z^T Q z + u^T R u) dt \right] \]

or for the stationary problem,

\[ J = E [ z^T Q z + u^T R u ] \]

The obvious solution

LQR + Kalman Filter

is correct! (But must prove this)
Conclusion: Separation principle always allows a controllable, observable system to have closed-loop poles at any desired location.

Note that there is no guarantee that

1. Compensator will be stable
2. System will be “robust”

Example $\dot{x} = -x + u$ \hspace{1cm} G(s) = \frac{1}{s+1}

$y = x$

Find compensator to place poles at $s = -3$, $s = -3$:

$a - kc = -3$

$= -1 - 1 \cdot k \Rightarrow k = 2$

$a - bf = -3$

$= -1 - 1 \cdot f \Rightarrow f = +2$
Compensator is

\[ K(s) = (a - bf - ke, \frac{1}{k}, -f) \]

\[ = (-5, 2, -2) \]

\[ = \frac{-4}{s+5} \]

Check:

\[ 1 - K(s)G(s) = 1 + \frac{4}{(s+1)(s+5)} \]

\[ \Rightarrow \Delta(s) = (s+1)(s+5) + 4 \]

\[ = s^2 + 6s + 9 = (s+3)^2 \checkmark \]
LQG Solution (Informal)

What is additional cost due to making an error in control?

\[ J = (x^T Q x + u^T R u) \, dt + \]
\[ J^*(x(t+dt), t+dt) \]
\[ = (x^T Q x + u^T R u) \, dt + x^T P x \]
\[ + x^T P x \, dt + x^T (A x + B u) \, dt \]
\[ + (A x + B u)^T P x \, dt \]

Let \( u = u^* + \delta u = -R^{-1}B^T P x + \delta u \). Then

\[ J = \delta u^T R \delta u \, dt + x^T P x = \delta u^T R \delta u \, dt + J^* \]

Therefore, the additional cost is

\[ \text{extra cost} = \int E(\delta u^T R \delta u) \, dt \]
\[ = \int \text{tr} \left[ \begin{array}{c} RF \Sigma F^T \end{array} \right] \, dt \]
\[ \text{since} \ \ \delta u = -E(x-x) = E \Sigma K E \]
This is the additional cost due to estimation error. The additional cost due to process noise is

\[ \int E \left[ w^T G^T P G w \right] dt \]

\[ = \int \text{tr} \left[ G^T P G w \right] dt \]

Therefore, the LQG cost is

\[ J = \int_0^T \text{tr} \left[ R K^T K \right] dt \]

Can be shown that also

\[ J = \int_0^T \text{tr} \left[ \Sigma E^T Q E + P L V L^T \right] dt \]

This follows from duality.