Random Processes

A better model for a dynamic system:

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) + G(t)w(t) \]
\[ y(t) = C(t)x(t) + v(t) \]

\( w(t) = \text{"process noise"} \)
\( v(t) = \text{"measurement noise"} \)

For example,

Plant = aircraft
\( w(t) = \text{turbulence} \)
\( v(t) = \text{gyro noise} \)

\( w(t), v(t) \) are random processes

A random process \( w(t) \) is described by its joint probability density functions

\[ p(w; t) \]
\[ p(w_1, w_2; t_1, t_2) \]
\[ p(w_1, w_2, w_3; t_1, t_2, t_3) \]
\[ : \]
E.g.,

\[ p(w_1, w_2; t_1, t_2) \Delta w_1 \Delta w_2 \]

\[ = \text{Prob} [ w_1 < w(t_1) \leq w_1 + \Delta w_1, w_2 < w(t_2) \leq w_2 + \Delta w_2 ] \]

In most cases, we care only about first and second order statistics:

mean:

\[ \bar{w}(t) = E[w(t)] = \int_{-\infty}^{\infty} w \ p[w; t] \ dw \]

mean square:

\[ E[w^2(t)] = \int_{-\infty}^{\infty} w^2 \ p[w; t] \ dw \]

Variance:

\[ \sigma^2(t) = E[(w(t) - \bar{w}(t))^2] \]

Correlation function:

\[ \rho(t) = E[w(t)w(t')] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} w_1 w_2 \ p[w_1, w_2; t_1, t_2] \ dw_1 \ dw_2 \]

For vector processes,

\[ R(t, t') = E[w(t)w^T(t')] \]

= "correlation matrix"

\[ R(t, t) = E[w(t)w^T(t)] = "covariance\ matrix" \]
Stationary processes

A process is stationary (in the strict sense) if

\[ p[\omega; t + \tau] = p[\omega; t] \]

\[ p[\omega_1, \omega_2; t_1 + \tau, t_2 + \tau] = p[\omega_1, \omega_2; t_1, t_2] \]

\[ \vdots \]

Stationary processes are to random processes as LTI systems are to linear systems.

For a stationary process, can define the power spectral density

\[ S_w(\omega) = \int_{-\infty}^{\infty} p(t) e^{-j\omega t} \, dt \]

\[ = \mathcal{F}[p(t)] \]

Conversely,

\[ p(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{j\omega t} \, d\omega \]

\[ p(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) \, d\omega = E[\omega^2|t] \]

= area under \( s(\omega) \)
White Noise

A white noise process \( w(t) \) is a random process with zero mean and flat power spectrum:

\[
S_w(\omega) = W = \text{const.}
\]

\[
\Rightarrow \rho(\tau) = W \delta(\tau)
\]

Note that \( \rho(0) = \infty \Rightarrow \]

\[
E[w^2(t)] = \infty
\]

So the process is not physically possible. But it is useful mathematically.

More generally, a non-stationary white noise process satisfies

\[
E[w(t)] = 0
\]

\[
E[w(t)w(t')] = \rho(t,t') = W(t) \delta(t-t')
\]

So don't have to stick to stationary processes.
Response of a linear system to white noise.

Consider the system

\[ \dot{x} = Ax + Gw, \quad x(0) = 0 \]

where \( w(t) \) is a zero-mean, white noise process with

\[ \mathbb{E}[w(t)w^T(\tau)] = R(t)\delta(t-\tau) \]

Statistics of \( x(t) \):

\[ x(t) = \int_0^t \Phi(t,\tau)G(\tau)w(\tau)\,d\tau \]

\[ \mathbb{E}[x(t)] = \mathbb{E}\left[ \int_0^t \Phi(t,\tau)G(\tau)w(\tau)\,d\tau \right] \]

\[ = 0 \]

\[ \mathbb{E}\left[ x(t_1) x^T(t_2) \right] \]

\[ = \mathbb{E}\left[ \int_0^{t_1} \Phi(t_1,\tau)G(\tau)w(\tau)\,d\tau \int_0^{t_2} \Phi^T(\tau_2)G^T(\tau_2)w(\tau_2)\,d\tau_2 \right] \]

\[ = \int_0^{t_1} \int_0^{t_2} \Phi(t_1,\tau_1)G(\tau_1)R(\tau_1)\delta(\tau_1-\tau_2)G^T(\tau_2)\Phi^T(t_2,\tau_2) \, d\tau_1 \, d\tau_2 \]
\[ P(t_1, t_2) = E \left[ x(t_1) x^T(t_2) \right] \]
\[ = \int_{\min(t_1, t_2)}^{\text{min}(t_1, t_2)} \Phi(t_1, \tau) \Phi(t_2, \tau) R(\tau) \Phi^T(t_2, \tau) \Phi^T(t_1, \tau) d\tau \]

**Take** \( t_1 > t_2 \). **Then**

\[ P(t_1, t_2) = \int_{0}^{t_2} \Phi(t_1, \tau) \Phi(t_2, \tau) R(\tau) \Phi^T(t_2, \tau) \Phi^T(t_1, \tau) d\tau \]
\[ = \Phi(t_1, t_2) \int_{0}^{t_2} \Phi(t_2, \tau) R(\tau) \Phi^T(t_2, \tau) \Phi^T(t_1, \tau) d\tau \]
\[ = \Phi(t_1, t_2) P(t_2, t_2) \]

Likewise, for \( t_1 < t_2 \),

\[ P(t_1, t_2) = P(t_1, t_1) \Phi^T(t_2, t_1) \]

*Typical (scalar) \( P(t_1, t_0) \)*
Usually, care about covariance of $x(t)$.

$P(t) = P(t, t)$  \(\nearrow\) looks like controllability!

\[
= \int_0^t \phi(t, \tau) g(\tau) R(\tau) g^T(\tau) \phi^T(t, \tau) \, d\tau
\]

Can easily find d.e. for $P(t)$:

\[
\dot{P}(t) = \int_0^t \phi(t, \tau) g R g^T \phi^T \, d\tau + \int_0^t \phi(t, \tau) g R g^T \phi^T \, d\tau
\]

\[+ gRg^T\]

\[
= \int_0^t A \phi(t, \tau) g R g^T \phi^T \, d\tau + \int_0^t \phi(t, \tau) g R g^T \phi^T A^T \, d\tau + gRg^T
\]

\[
\Rightarrow
\dot{P}(t) = A(t) P(t) + P(t) A^T(t) + g(t) R(t) g^T(t)
\]

= "Lypunov Equation"