Unit M4.4
Simple Beam Theory

Readings:
CDL 7.1 - 7.5, 8.1, 8.2
LEARNING OBJECTIVES FOR UNIT M4.4

Through participation in the lectures, recitations, and work associated with Unit M4.4, it is intended that you will be able to……..

• …..**describe** the aspects composing the model of a beam associated with deformations/displacements and stresses (i.e. Simple Beam Theory) and **identify** the associated limitations

• ….**apply** the basic equations of elasticity to **derive** the solution for the general case

• ….**identify** the beam parameters that characterize beam behavior and **describe** their role
We have looked at the statics of a beam, but want to go further and look at internal stress and strain and the displacement/deformation. This requires a particular model with additional assumptions besides those on geometry of “long and slender.”

**Figure M4.4-1  Geometry of a beam**

h and b are “encompassing/extreme” dimensions still have:  \( L >> h, b \)

Now also consider
Assumptions on Stresses

We have said that loading is in the plane x-z and is transverse to the long axis (the x-axis)
The first resulting assumption from this is:

All loads in y - direction are zero
⇒ all stresses in y-direction are zero:

$$\sigma_{yy} = \sigma_{xy} = \sigma_{yz} = 0$$

--> Next, we “assume” that the only significant stresses are in the x-direction.

⇒ $$\sigma_{xx}, \sigma_{xz} \gg \sigma_{zz}$$

--> Why (valid)? Look at isolated element and moment equilibrium

*Figure M4.4-2*  Illustration of moment equilibrium of “isolated element” of beam
\[ \sum M_0 \text{ (magnitude)} \Rightarrow \sigma_{zz} \text{ (moment arm)} + \sigma_{xx} \text{ (moment arm)} = 0 \]

moment arm for \( \sigma_{zz} \approx L \); moment arm for \( \sigma_{xx} \approx h \)

but, \( h << L \Rightarrow \sigma_{xx} \gg \sigma_{zz} \)

can make same argument for \( \sigma_{xz} \)

--- thus, assumption is only non zero stresses are \( \sigma_{xz} \) and \( \sigma_{xx} \)

\[ \Rightarrow \sigma_{zz} \approx 0 \]

To complete our model, we need....

Assumptions on Deformations

The key here is the “Bernouilli-Euler Hypothesis” (~1750):

“Plane sections remain plane and perpendicular to the midplane after deformation”

--- To see what implications this has, consider an infinitessimal element that undergoes bending (transverse) deformation:
Figure M4.4-3  Basic deformation of infinitessimal element to beam according to “plane sections remains plane” (Bernouilli-Euler Hypothesis)

Define: \( w = \) deflection of midplane/midline (function of \( x \) only)
Use geometry to get deflection in x-direction, $u$, of point $q$ ($q$ to $Q$)

*Figure M4.4-4*  Local geometry of deflection of any point of beam

- Figure showing local geometry of deflection with $\phi$, $Q$, $P$, $u$, and $\partial w/\partial x$.
- $\phi = \tan^{-1} \left( \frac{dw}{dx} \right)$
- $u = -z \sin \phi$
- Distance of $q$ above $x$ direction is opposite to $x$-direction.

If deformations/angles are small: $\sin \phi \approx \phi$; $\phi \approx \frac{\partial w}{\partial x}$

$\Rightarrow u \approx -z \phi$
Thus, implication of assumption on displacement is:

$$u(x, y, z) = -z \frac{dw}{dx} \quad (1)$$

$$v(x, y, z) = 0 \quad \text{(nothing in y-direction)}$$

$$w(x, y, z) = w(x) \quad (2) \quad \text{(cross-section deforms as a unit) } \Rightarrow \text{ (plane sections remain plane)}$$

We have all the necessary assumptions as we have the structural member via assumptions on geometry, stress, and displacements/deformations. We now use the Equations of Elasticity to get the….

Resulting Equations

--> First apply the Strain-Displacement Equations….
\[ \varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{d^2 w}{dx^2} \quad (3) \]
\[ \varepsilon_{yy} = \frac{\partial v}{\partial y} = 0 \quad \varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \]
\[ \varepsilon_{xy} = \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) = 0 \]
\[ \varepsilon_{yz} = \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) = 0 \]
\[ \varepsilon_{xz} = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) = \frac{1}{2} \left( -\frac{dw}{dx} + \frac{dw}{dx} \right) = 0 \]

\[ \Rightarrow \text{This is consistent with assumption by B-E (no shearing gives plane sections remain plane and perpendicular)} \]

\[ \Rightarrow \text{Next use stress-strain. We'll go to orthotropic as most general we can do} \]
\[ \varepsilon_{xx} = \frac{\sigma_{xx}}{E_x} \]  \hspace{1cm} (4)

\[ \begin{align*}
\varepsilon_{yy} &= -\nu_{yy} \frac{\sigma_{xx}}{E_x} \\
\varepsilon_{zz} &= -\nu_{zz} \frac{\sigma_{xx}}{E} \\
\end{align*} \]

Note: "slight" inconsistency between assumed displacement state and those resulting strains, and the resulting strains from the stress-strain equations

\[ \begin{align*}
\varepsilon_{xy} &= \frac{\sigma_{xy}}{2G_{xy}} = 0 \\
\varepsilon_{yz} &= \frac{\sigma_{yz}}{2G_{yz}} = 0 \\
\varepsilon_{xz} &= \frac{\sigma_{xz}}{2G_{xz}} \neq 0 \\
\end{align*} \]

Note: again a "slight" inconsistency
We “get around” these inconsistencies by saying that $\varepsilon_{yy}$, $\varepsilon_{zz}$, and $\varepsilon_{xz}$ are very small but not quite zero. This is an approximation (part of model). Will check this later.

--> Finally use the Equilibrium Equations:

Assumption: no body forces ($f_i = 0$)

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{zx}}{\partial z} = 0 \quad (5)
\]

\[
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} = 0 \quad \Rightarrow \quad 0 = 0
\]

\[
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \quad (6)
\]

--> So we have 5 unknowns: w, u, $\varepsilon_{xx}$, $\sigma_{xx}$, $\sigma_{xz}$

(Note: $\sigma_{zz}$ is ignored)

--> And we have 5 equations: 1 from geometry: (1)

1 from strain-displacement: (3)

1 from stress-strain: (4)

2 from equilibrium: (5), (6)
So then we have the right number of equations for the number of unknowns. So we consider the:

**Solution: Stresses and Deflections**

In doing this, it is first important to relate the point-by-point stresses to the average internal forces (F, S, M).

To do this, consider a cut face (do here for rectangular cross-section; will generalize later)

*Figure M4.4-5 Geometry of Equilibrium via stresses on cut face of beam*
Equilipollence (i.e., equally powerful) shows: (no variation in y)

\[ F = \int_{-h/2}^{h/2} \sigma_{xx} \, b \, dz \quad (7) \]

\[ S = -\int_{-h/2}^{h/2} \sigma_{xz} \, b \, dz \quad (8) \]

\[ M = -\int_{-h/2}^{h/2} \sigma_{xx} \, b \, z \, dz \quad (9) \]

\[ \sigma_{xx} = E_x \, \epsilon_{xx} = -E_x \, z \, \frac{d^2 w}{dx^2} \quad (10) \]

Now put this in (7):

\[ F = -E_x \, \frac{d^2 w}{dx^2} \int_{-h/2}^{h/2} z \, b \, dz \]

\[ = -E_x \, \frac{d^2 w}{dx^2} \frac{z^2}{2} \, b \bigg|_{-h/2}^{h/2} = 0 \quad \text{since no axial force in pure beam case} \]
(Note: something that carries axial and bending forces is known as a beam-column/rod)

we also place the result for $\sigma_{xx}$ (10) in the equation for the internal moment (9):

$$M = E_x \frac{d^2 w}{dx^2} \int_{-h/2}^{h/2} z^2 bdz$$

we define:

$$I = \int_{-h/2}^{h/2} z^2 b dz$$

units of $[L^4]$ = Area (Second) Moment of Inertia of beam cross-section [about y-axis]

Note: For rectangular cross-section

$$I = \frac{bh^3}{12}$$

--> will look at this further in next unit
This results in the following:

\[
M = E_x I \frac{d^2 w}{dx^2} \tag{12}
\]

**Moment-Curvature relation** for beam

Note: EI is controlling parameter - “flexural rigidity” or “bending stiffness”. Has:
- geometrical contribution, I
- material contribution, E
- units: \([F \cdot L] = \left[ \frac{F}{L^2} \right] \left[ L^4 \right] \left[ \frac{L}{L^2} \right]\)

\[\frac{\partial \sigma_{zx}}{\partial z} = - \frac{\partial \sigma_{xx}}{\partial x} \tag{5}\]

Multiply each side by b and integrate from z to h/2 to get:
\[
\int_{z}^{h/2} b \frac{\partial \sigma_{zx}}{\partial z} \, dz = -\int_{z}^{h/2} \frac{\partial \sigma_{xx}}{\partial x} \, b \, dz
\]

First take (12) and put it in (10):

\[
\sigma_{xx} = -E_x z \frac{d^2w}{dx^2} = -E_x z \frac{M}{E_x I}
\]

\[
\Rightarrow \sigma_{xx} = -\frac{Mz}{I} \quad \text{(13)}
\]

Units: $\begin{bmatrix} F \\ L^2 \end{bmatrix} = \left[ \frac{FL}{L^4} \right]$  

Now, work on integrating the pending equation:

\[
\Rightarrow b \sigma_{xz}(z) \bigg|_{z}^{h/2} = -\int_{z}^{h/2} -\frac{\partial M}{\partial x} \, \frac{zb}{I} \, dz
\]

Recall that: \( \frac{dM}{dx} = S \) to get:
\[
\Rightarrow b \left[ \sigma_{xz} \left( \frac{h}{2} \right) - \sigma_{xz}(z) \right] = + \int_{z}^{h/2} S \frac{zb}{I} \, dz
\]

Note that the \( \sigma_{xz} \) at the top of surface is zero.

Also define:

\[
Q = \int_{z'}^{h/2} zbdz = \text{(first) Moment of area about the center}
\]

So:

\[
\sigma_{xz}(z) = - \frac{SQ}{Ib}
\]

**shear stress-Shear relation**

Units:

\[
\left[ \frac{F}{L^2} \right] = \left[ \frac{F}{L^4} \right] \left[ \frac{L^3}{L} \right]
\]
For a rectangular section:

For a rectangular section:

**Figure M4.4-6** Geometry for assessing (first) moment of area about centerline

\[ Q = \int_{z'}^{h/2} zbdz \]

\[ = \left. \frac{z^2}{2}b \right|_{z'}^{h/2} = \frac{b}{2} \left[ \frac{h^2}{4} - z'^2 \right] \]

(maximum at \( z' = 0 \), the centerline)

---> Again, will look at this further and generalize in the next unit
The summary of how we can solve for the stress/strain/displacement states in a beam is presented in handout M-5.

In the next section, we look at what this solution generally means and examine it for various situations.
Unit M4.4 (New) Nomenclature

EI -- flexural rigidity or boundary stiffness of beam cross-section
I -- Area (Second) Moment of Inertia of beam cross-section (about y-axis)
Q -- (First) Moment of area above the centerline
u -- deflection of point of beam in x-direction
v -- deflection of point of beam in y-direction
w -- deflection of (midpoint/midline of) beam in z-direction
φ -- slope of midplane of beam at any point x ( = dw/dx)
d²w/dx² -- curvature of beam (midplane/midline) at any point x of beam
σxx -- beam bending stress
σxz -- beam transverse shear stress