18.06, Fall 2003, Problem Set 3 Solutions

1. 
(a) 
\[ U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & 0 \end{bmatrix}. \]

(b) 
\[ L = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, U = \begin{bmatrix} -1 & 3 & 5 \\ 0 & 0 & -3 \end{bmatrix}. \]

2. 
If \( Ux = 0 \), then \( LUx = L(Ux) = L \cdot 0 = 0 \).

If \( LUx = 0 \), then \( L^{-1}LUx = L^{-1}0 = 0 \), so \( Ux = 0 \). (Another proof: Since \( L \) is invertible, \( Ly = 0 \) implies \( y = 0 \), so \( L(Ux) = 0 \) implies \( Ux = 0 \).)

3. 
Here is one possibility:
\[ \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}. \]

4. 
The nullspace is the set of vectors orthogonal to \( v = (-1, 1, 2) \).

There are two ways to see this.

\( v^T x \) is simply the dot product of \( v \) with \( x \). Therefore, \( vv^T x = (v \cdot x)v \).

(Multiplication of a scalar by a vector is commutative.) If \( v \) and \( x \) are orthogonal, then the dot product is 0, and hence \( vv^T x = (v \cdot x)v \) is the zero vector. If \( v \) and \( x \) are not orthogonal, then the dot product is nonzero, so \( vv^T x = (v \cdot x)v \) is some nonzero vector in the direction of \( v \).
The other argument is to perform Gaussian elimination on

\[ vv^T = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}, \]

finding that the nullspace consists of the solutions to the equation \(-x_1 + x_2 + 2x_3 = 0\), i.e. \(v_1x_1 + v_2x_2 + v_3x_3 = v \cdot x = 0\).

5.

(a) \(b = (0,0)\) is the only vector for which the solution set is a subspace of \(\mathbb{R}^2\). When \(b\) is the zero vector, the solution set is the nullspace, which is proven to be a subspace in the book. When \(b\) is not the zero vector, \((0,0,0)\) is not a solution of the equation, and hence the solution set is not a subspace.

(b) Yes, the set of vectors \(b\) for which there is a solution forms a subspace of \(\mathbb{R}^2\). It suffices to check that this set \(S\) is closed under scalar multiplication and addition: If \(Ax^{(1)} = b^{(1)}\) and \(Ax^{(2)} = b^{(2)}\), then \(A(cx^{(1)}) = cb^{(1)}\) for any scalar \(c\), and \(A(x^{(1)} + x^{(2)}) = b^{(1)} + b^{(2)}\). Therefore, \(cb^{(1)}\) and \(b^{(1)} + b^{(2)}\) are also in the set \(S\).

6.

Yes, the set of polynomials is a subspace of \(F\). Check closure under addition and scalar multiplication as follows. Writing \(p(x) = a_nx^n + \cdots + a_1x + a_0\) and \(q(x) = b_mx^m + \cdots + b_1x + b_0\), it can be shown quickly that \(\alpha p(x)\) is a polynomial for any \(\alpha \in \mathbb{R}\), and \(p(x) + q(x)\) is also a polynomial.

7.

(a) If \(x\) is the rotation by \(\theta_1\) radians and \(y\) is the rotation by \(\theta_2\) radians, then \(x + y\) can be defined as the rotation by \(\theta_1 + \theta_2\) radians.

(b) \(-x\) can be defined as the rotation by \(-\theta_1\) radians, i.e. a clock-wise rotation of \(\theta_1\) radians.

More generally, multiplying a rotation of \(\theta\) radians by a scalar \(\alpha\) should result in a rotation of \(\alpha\theta\) radians. (A negative rotation should be considered a clock-wise rotation.) To check that this really defines a vector space, you would need to check the eight conditions on page 118.
8.

(a) Let \((b_1, b_2, b_3)\) be any vector in the column space. Then

\[
\begin{bmatrix}
1 & 3 & 1 & b_1 \\
3 & 8 & 2 & b_2 \\
2 & 4 & 0 & b_3 \\
\end{bmatrix}
\]

reduces to

\[
\begin{bmatrix}
1 & 3 & 1 & b_1 \\
0 & -1 & -1 & -3b_1 + b_2 \\
0 & -2 & -2 & -2b_1 + b_3 \\
\end{bmatrix}
\]

and then

\[
\begin{bmatrix}
1 & 3 & 1 & b_1 \\
0 & -1 & -1 & -3b_1 + b_2 \\
0 & 0 & 0 & 4b_1 - 2b_2 + b_3 \\
\end{bmatrix}
\].

As long as \(4b_1 - 2b_2 + b_3 = 0\), the system is solvable. The column space is precisely the set of vectors \((b_1, b_2, b_3)\) for which the system \(Ax = b\) is solvable.

(b) If we multiply the first row by \(c_1\), the second row by \(c_2\), and the third row by \(c_3\), then the combination gives the zero vector if and only if \((c_1, c_2, c_3)\) is a scalar multiple of \((4, -2, 1)\). We can think of solving

\[
\begin{bmatrix}
c_1 & c_2 & c_3
\end{bmatrix} A = 0,
\]

or

\[
A^T \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = 0,
\]

i.e. we are looking at the nullspace of \(A^T\).