18.06, Fall 2003, Problem Set 5 Solutions

1. Let \( A = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \)

(a) If \( a_1, a_2, \) and \( a_3 \) are orthonormal, then \( A^T A \) is the identity matrix. Thus

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = A^T b,
\]

so \( x_1 = a_1^T b. \)

(b) If \( a_1, a_2, \) and \( a_3 \) are orthogonal, then \( A^T A \) is a diagonal matrix where the \( i \)th entry is \( a_i^T a_i. \) \( A^T A x = A^T b, \) so

\[
\begin{bmatrix}
  a_1^T a_1 x_1 \\
  a_2^T a_2 x_2 \\
  a_3^T a_3 x_3
\end{bmatrix} = A^T b,
\]

and so \( x_1 = \frac{a_1^T b}{a_1^T a_1}. \)

(c) If \( a_1, a_2, \) and \( a_3 \) are independent, then \( A \) is invertible, so \( x_1 \) is the first component of \( A^{-1} b. \)

2. (a) We want to find the nullspace of the matrix \( \begin{bmatrix} 2 & -1 & 4 \end{bmatrix} \), and this has the basis

\[
\left\{ \begin{bmatrix} 1 \\
  2 \\
  0
\end{bmatrix}, \begin{bmatrix} -2 \\
  0 \\
  1
\end{bmatrix} \right\}
\]

(Note: any basis equivalent to this one is acceptable.)

(b) Applying the Gram-Schmidt process to this basis, we obtain the orthonormal basis

\[
\left\{ \begin{bmatrix} 1/\sqrt{5} \\
  2/\sqrt{5} \\
  0
\end{bmatrix}, \begin{bmatrix} -8/\sqrt{105} \\
  4/\sqrt{105} \\
  5/\sqrt{105}
\end{bmatrix} \right\}
\]

(Note: equivalent answers are acceptable.)

(c) If we let \( b = \begin{bmatrix} 0 \\
 -3 \\
  1
\end{bmatrix} \) and the subspace \( V \) be the plane \( 2x - y + 4z = 0, \) then we want to project \( b \) onto \( V \) to get the vector \( p, \) and then we want the length of the vector \( b - p. \) We know a basis for this subspace from the previous part of the problem (or use the basis found in the first part of the problem, but the orthonormal basis makes some of the calculations easier), so form the matrix \( A \) whose columns are this basis:
\[
A = \begin{bmatrix}
1/\sqrt{5} & -8/\sqrt{105} \\
2/\sqrt{5} & 4/\sqrt{105} \\
0 & 5/\sqrt{105}
\end{bmatrix}.
\]

Recall that \( p = A(A^T A)^{-1} A^T b \). Since the columns of \( A \) are orthonormal, \( A^T A \) is the 2-by-2 identity matrix. Thus \( p = A A^T b \).

\[
p = A A^T b = \frac{1}{21} \begin{bmatrix}
17 & 2 & -8 \\
2 & 20 & 4 \\
-8 & 4 & 5
\end{bmatrix} \begin{bmatrix}
0 \\
-3 \\
1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
-2 \\
-8 \\
1
\end{bmatrix}.
\]

Thus the error vector \( e = b - p = \begin{bmatrix}
2/3 \\
-1/3 \\
4/3
\end{bmatrix} \), so the distance between \( b \) and \( V \) is the length of \( e \), which is \( \sqrt{21}/3 \).

3. (a) Since the square of \( T \) is the identity transformation, \( T \) must be its own inverse. That means that the matrix \( A \) is equal to the matrix \( A^{-1} \).

(b) \( \det(A) = (\det(A^{-1}))^{-1} = (\det(A))^{-1} \), so \( \det(A) = \pm 1 \).

(c) Let \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

Consider first the case that \( \det(A) = 1 \). Then, since \( A = A^{-1} \), we know that \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \), so \( b = c = 0 \) and \( a = d \). The determinant then is \( a^2 \), so \( A \) is either \( I \) or \( -I \).

Now suppose \( \det(A) = -1 \). Then \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix} \), so \( a = -d \). Thus \( A \) can be any matrix of the form \( \begin{bmatrix} a & b \\ c & -a \end{bmatrix} \) satisfying \( a^2 + bc = 1 \).

4. (a) \[
A = \begin{bmatrix}
3 & -1 & 0 \\
-1 & 3 & -1 \\
0 & -1 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
3 & -1 & 0 \\
0 & 8/3 & -1 \\
0 & -1 & 3
\end{bmatrix} \rightarrow \begin{bmatrix}
3 & -1 & 0 \\
0 & 8/3 & -1 \\
0 & 0 & 21/8
\end{bmatrix}.
\]

Thus \( \det(A) = 3(8/3)(21/8) = 21 \).

(b) Since \( \det(A) \) is nonzero, \( A \) is invertible. \( \det(A^{-1}) = (\det(A))^{-1} = 1/21 \).

(c) \[
\det(A - \lambda I) = \det \begin{bmatrix}
3 - \lambda & -1 & 0 \\
-1 & 3 - \lambda & -1 \\
0 & -1 & 3 - \lambda
\end{bmatrix}
\]
\[
= (3 - \lambda) \det \begin{bmatrix}
3 - \lambda & -1 \\
-1 & 3 - \lambda
\end{bmatrix} - (-1) \det \begin{bmatrix}
-1 & -1 \\
0 & 3 - \lambda
\end{bmatrix} + 0 = (3 - \lambda)^3 - 2(3 - \lambda).
\]
(d) Certainly \( \lambda = 3 \) is one value that will make this determinant zero. If \( \lambda \neq 3 \), then we have \( 0 = (3 - \lambda)^2 - 2 = 7 - 6\lambda + \lambda^2 \), so \( \lambda = 3 \pm \sqrt{2} \).

5. Expanding by cofactors along the first column, we have

\[
\det A = c \cdot \det \begin{bmatrix} 1 & -3 \\ 1 & 6 \end{bmatrix} - 0 + (-1) \det \begin{bmatrix} 6 & c^2 \\ 1 & -3 \end{bmatrix}.
\]

Thus \( \det A = c^2 + 9c + 18 = (c + 3)(c + 6) \), so \( A \) is a singular matrix when \( c = -3 \) or \( -6 \).

6. (a) There are \( 4! = 24 \) permutations of four elements, because there are four choices for the first element, three choices for the second element, two choices for the third element, and one choice for the last element in the permutation.

(b) The even permutations are \((1, 2, 3, 4), (2, 1, 4, 3), (3, 4, 1, 2), (4, 3, 2, 1), (2, 3, 1, 4), (3, 1, 2, 4), (2, 4, 3, 1), (4, 1, 3, 2), (3, 2, 4, 1), (4, 2, 1, 3), (1, 3, 4, 2), \) and \((1, 4, 2, 3)\).

The odd permutations are \((2, 1, 3, 4), (3, 2, 1, 4), (4, 2, 3, 1), (1, 3, 2, 4), (1, 4, 3, 2), (1, 2, 4, 3), (2, 3, 4, 1), (3, 4, 2, 1), (4, 1, 2, 3), (2, 4, 1, 3), (3, 1, 4, 2), \) and \((4, 3, 1, 2)\).

7. Subtracting 1 from the \( n,n \) entry subtracts its cofactor once from the determinant. Since its cofactor is the determinant of a smaller Pascal matrix, this value is 1, so the determinant of the new matrix becomes zero.