18.06, Fall 2003, Problem Set 6

Due before 4PM on Wednesday November 12, 2003, in the boxes in 2-106. No late homework will be accepted. There is one box for each recitation section. For full credit, please be sure to show and explain your work.

1. Consider the matrix:

   \[
   A = \begin{bmatrix}
   0.5 & b & 0 & a \\
   a & 0.5 & b & 0 \\
   0 & a & 0.5 & b \\
   b & 0 & a & 0.5 \\
   \end{bmatrix}
   \]

   (a) For which values of \(a\) and \(b\) is \(A\) Markov?

   (b) Verify that \((1,1,1,1)\) is an eigenvector of \(A\). What is the corresponding eigenvalue? Is your result consistent with your answer to part 1a?

   (c) Let \(\omega\) be a complex number such that \(\omega^4 = 1\). What are all possible values for \(\omega\)?

   (d) Verify that \((1,\omega, \omega^2, \omega^3)\) is an eigenvector whenever \(\omega^4 = 1\). What are the corresponding eigenvalues?

   (e) Give a formula for the determinant of \(A\) in terms of \(a\) and \(b\).

   (f) Verify your answers to (1d) and (1e) by using MATLAB. Take \(a = 1\), \(b = 0\) and compute using MATLAB the eigenvalues of \(A\) and \(\det(A)\). Do these match your answers to (1d) and (1e)? Now take \(a = 0.25\) and \(b = 0.25\) and compute the eigenvalues of \(A\) and \(\det(A)\). Do these match your answers to (1d) and (1e)? Show your MATLAB output.

   (g) Take \(a = 0.3\) and \(b = 0.2\). What does \(A^k\) tend to as \(k\) tends to infinity?

   (h) Give conditions on \(a\) and \(b\) such that \(A^k\) bounded for all \(k\) (i.e. the entries of \(A^k\) do not become arbitrarily large).

   (i) Can \(A\) be diagonalized for all values of \(a\) and \(b\)?

   (j) Let \(B = e^A\). What is \(\det(B)\)? (Hint: Look at the formula for \(e^A\) based on a diagonalization of \(A\), at least for those values of \(a\) and \(b\) for which \(A\) can be diagonalized.) Your answer should be independent of \(a\) and \(b\).

2. Let \(P\) be a projection matrix, corresponding to projecting \(\mathbb{R}^n\) onto a subspace \(F\) of dimension \(k\).

   (a) What are the eigenvalues of \(P\)? Give them with their multiplicities.

   (b) Argue that \(P\) is always diagonalizable. (Either remember all cases we have seen that are diagonalizable, or construct a diagonalization.)

   (c) Using the Taylor series definition of a matrix exponential, give a simple formula for \(e^{Pt}\), just in terms of \(P\) and \(t\).

   (d) To check the formula you just got, do the following in MATLAB. Select any vector \(a\) in \(\mathbb{R}^3\) and compute the projection matrix \(P\) corresponding to projecting onto the line defined by \(a\). Then compute \(expm(P)\) and check that this agrees to your formula for \(t = 1\). Also, observe that \(expm(P)\) is not simply obtained by taking the exponential of every entry (which in MATLAB is done by typing \(exp(P)\)). Give the commands you typed and the results you got.

   (e) Consider the differential equation \(\frac{\text{d}u}{\text{d}t} = Pu\), where \(P\) again is a projection matrix. Is this differential equation stable? Does it blow up?