18.06, Fall 2003, Problem Set 8

Due before 4PM on Wednesday November 26, 2003, in the boxes in 2-106. No late homework will be accepted. There is one box for each recitation section. For full credit, please be sure to show and explain your work.

1. Suppose $A$ is a $3$ by $3$ symmetric matrix with unit eigenvectors $u_1$, $u_2$, and $u_3$. If its eigenvalues are $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = -1$, what are the matrices $U, \Sigma$, and $V^T$ in its SVD?

To find the columns of $V$, we determine the eigenvectors of $A^T A$. Since $A$ is symmetric, $A^T A = A^2$. The eigenvalues of $A^2$ are $4, 1, 1$ with corresponding eigenvectors $u_1$, $u_2$, and $u_3$, which are orthonormal since they are unit vectors and correspond to distinct eigenvalues of $A$. Hence, $V = [u_1 \ u_2 \ u_3]$.

Now, the singular values are the square roots of the eigenvalues of $A^T A$. So, $\Sigma = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Finally, the matrix $U$ is $[u_1 \ u_2 - u_3]$ since $AV = U \Sigma$.

2. Let $J = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $K = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

(a) What are the eigenvalues of $J$ and $K$?

Each of $J$ and $K$ has one eigenvalue, 0, with (algebraic) multiplicity four. (The geometric multiplicity of the eigenvalue 0 is 2 in both cases.)

(b) Show that $J$ is not similar to $K$. There are several ways of showing this.

Suppose there exists an invertible matrix $M = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix}$ such that

$J = M^{-1} KM$. Hence, we must have $MJ = KM$. This gives

$\begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix} = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Thus, $m_{21} = m_{22} = m_{11} = m_{41} = m_{42} = m_{31} = 0$. Hence, the first column of $M$ is zero, contradicting that $M$ is invertible. So $J$ cannot be similar to $K$.

Here is another way. If $J$ and $K$ were similar, then so would be $J^2$ and $K^2$. But $J^2$ is not 0 while $K^2$ is 0. This means that the eigenvalue 0 has geometric multiplicity 4 in $K^2$ and less than 4 in $J^2$, and this contradicts the fact that they are similar.

3. For each of the following statements, state whether the statement is true or false. If the statement is true, explain why it is true. If the statement is false, give a counterexample to the statement (i.e. give an specific example for which the statement is incorrect and show
that the statement is false for that example).

(a) If \( A \) is similar to \( B \), then \( A^2 \) is similar to \( B^2 \).

(True) \( A = M^{-1}BM \) for some invertible \( M \). This gives \( A^2 = (M^{-1}BM)(M^{-1}BM) = M^{-1}B^2M \). So \( A^2 \) is similar to \( B^2 \).

(b) If \( A^2 \) is similar to \( B^2 \), then \( A \) is similar to \( B \).

(False) Consider \( A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \) and \( B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \). \( A^2 \) and \( B^2 \) are the zero matrix and so they are similar. But \( A \) and \( B \) are not similar. (See Example 2 on p. 344.) An even simpler example is just \([-1] \) and \([1]\).

(c) \( \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \) is similar to \( \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} \).

(True) \( \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} = M^{-1} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix} M \) where \( M = \begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix} \).

(d) If we exchange rows 1 and 2 of \( A \), and then exchange columns 1 and 2, the eigenvalues stay the same.

(True) The matrix is \( B = P_{12}AP_{12} \). But \( P_{12} \) is its own inverse. So \( A \) is similar to \( B \). By result 6Q on p. 343, \( A \) and \( B \) have the same eigenvalues.

4. Suppose a linear transformation \( T \) transforms \((1,1)\) to \((2,2)\) and \((2,1)\) to \((0,0)\). Find \( T(v) \) when

(a) \( v = (5,3) \)

Since \((5,3) = (1,1) + 2(2,1)\) and \( T \) is linear, \( T(5,3) = T(1,1) + 2T(2,1) = (2,2) \).

(b) \( v = (0,1) \).

Since \((0,1) = 2(1,1) - (2,1)\) and \( T \) is linear, \( T(0,1) = 2T(1,1) - T(2,1) = (4,4) \).

5. Let \( V \) denote the vector space of all 2 by 2 matrices. Let \( T \) denote the function defined by

\[
T(M) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} M \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\]

for all \( M \) in \( V \).

(a) Show that \( T \) is a linear transformation.

Let \( A \) denote \( \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \) and \( B \) denote \( \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \). For any matrices \( U \) and \( W \) from \( V \) and scalars \( c \) and \( d \), we have

\[
T(cU + dW) = A(cU + dW)B = (cAU + dAW)B = cAUB + dAWB = cT(U) + dT(W).
\]

So \( T \) satisfies the linearity condition and so it is a linear transformation.
(b) What is the dimension of the range of $T$?

Let $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $T(M) = AMB = \begin{bmatrix} 0 & a + b + c + d \\ 0 & 0 \end{bmatrix}$. Since $a + b + c + d$ is arbitrary, the dimension of the range of $T$ is 1.

(c) Describe all matrices in the kernel of $T$.

The matrices in the kernel of $T$ are precisely the matrices $M$ such that $T(M)$ is zero. But from the previous part, we see that $T(M)$ is zero if and only if the sum of all the entries of $M$ is zero. So a matrix in $V$ is in the kernel of $T$ if and only if the sum of all its entries is zero.