18.06, Fall 2003, Problem Set 9

Due before 4PM on Wednesday, December 3rd, 2003, in the boxes in 2-106. No late homework will be accepted. There is one box for each recitation section. For full credit, please be sure to show and explain your work.

1. Let \( \mathbf{x}_1, \mathbf{x}_2 \) be a basis of \( \mathbb{R}^2 \) and \( \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \) be a basis of \( \mathbb{R}^3 \). Let \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) be a linear transformation defined by

\[
f(\mathbf{y}_1) = \mathbf{x}_1 - \mathbf{x}_2, f(\mathbf{y}_2) = \mathbf{x}_2, f(\mathbf{y}_3) = -\mathbf{x}_1 - \mathbf{x}_2.\]

Find a new basis \( \mathbf{u}_1, \mathbf{u}_2 \) of \( \mathbb{R}^2 \) (written in terms of \( \mathbf{x}_1 \) and \( \mathbf{x}_2 \)) and a new basis \( \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \) of \( \mathbb{R}^3 \) (written in terms of \( \mathbf{y}_1, \mathbf{y}_2, \) and \( \mathbf{y}_3 \)) such that

\[
f(\mathbf{v}_1) = 3\mathbf{u}_1, f(\mathbf{v}_2) = 2\mathbf{u}_2, f(\mathbf{v}_3) = 0.\]

2. Let

\[
A = \begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}.
\]

(a) Compute the Singular Value Decomposition (SVD) of \( A \) and \( B \).
(b) Compute the pseudo-inverse \( A^+ \) and \( B^+ \) of \( A \) and \( B \), respectively.

3. If \( P \) and \( Q \) are orthogonal, show that \( A \) and \( PAQ \) have the same singular values.

4. Compute the Fourier coefficients \( a_k \) and \( b_k \) of \( f(x) = \frac{1}{2} \sin^2(x) + x \) defined between 0 and 2\( \pi \).

5. True or false (give a reason if true or a counterexample if false):

(a) If \( A \) is a real matrix then \( A + iI \) is invertible.
(b) If \( A \) is a Hermitian matrix then \( A + iI \) is invertible.
(c) If \( U \) is a unitary matrix then \( A + iI \) is invertible.
(d) The inverse of a Hermitian matrix is Hermitian.
(e) If \( U \) and \( V \) are unitary matrices then \( UV \) is unitary.
(f) If \( U \) and \( V \) are unitary matrices then \( U + V \) is unitary.