Your PRINTED name is: **SOLUTIONS**

Please circle your recitation:

1) M 2 2-131 P. Lee 2-087 2-1193 lee
2) M 2 2-132 T. Lawson 4-182 8-6895 tlawson
4) T 10 2-132 P.-O. Persson 2-363A 3-4989 persson
5) T 11 2-131 P.-O. Persson 2-363A 3-4989 persson
6) T 11 2-132 P. Pylyavskyy 2-333 3-7826 pasha
7) T 12 2-132 T. Lawson 4-182 8-6895 tlawson
8) T 12 2-131 P. Pylyavskyy 2-333 3-7826 pasha
9) T 1 2-132 A. Chan 2-588 3-4110 alicec
10) T 1 2-131 D. Chebikin 2-333 3-7826 chebikin
11) T 2 2-132 A. Chan 2-588 3-4110 alicec
12) T 3 2-132 T. Lawson 4-182 8-6895 tlawson
1 (30 pts.) Suppose $A$ is $m$ by $n$ with **linearly dependent columns**. Complete with as much true information as possible:

(a) The rank of $A$ is . . .

at most $n - 1$ (and at most $m$, which is a stronger statement if $m < n - 1$).

(b) The nullspace of $A$ contains . . .

at least one non-zero vector. (The dimension of the nullspace is $n$ minus the column rank of $A$, i.e., at least 1.)

(c) (more words needed) The equation $A^T y = b$ has no solution for some right hand sides $b$ because . . .

the rows of the matrix $A^T$, which are the same as the columns of $A$, are linearly dependent, so $A^T$ is not full row-rank. Thus the reduced row echelon form of $A^T$ contains a row of all zeroes, so the components of $b$ must satisfy a certain linear relation in order for $A^T y = b$ to have a solution.
2 \ (40 \text{ pts.}) \quad \text{Suppose } A \text{ is this } 3 \text{ by } 4 \text{ matrix:}

\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 \\
2 & 3 & 4 & 5 \\
3 & 4 & 5 & 6
\end{bmatrix}
\]

(a) A specific basis for the column space of \( A \) is \boxed{\text{ }}.

(b) For which vectors \( b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \) does \( Ax = b \) have a solution? Give conditions on \( b_1, b_2, b_3 \).

(c) There is no 4 by 3 matrix \( B \) for which \( AB = I \) (3 by 3). Give a good reason (is this because \( A \) is rectangular?).

(d) Find the complete solution to \( Ax = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \).

\[\text{Solution}\]

(a) \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \) is a basis for the column space of \( A \). So is \( \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \).

(b) Row reducing the augmented matrix \([A \ b]\), we get

\[
\begin{bmatrix}
1 & 2 & 3 & 4 & b_1 \\
0 & -1 & -2 & -3 & b_2 - 2b_1 \\
0 & 0 & 0 & 0 & b_3 - 2b_2 + b_1
\end{bmatrix}
\]

The linear equations corresponding to the top two rows can be satisfied regardless of the values of \( b_1 \) and \( b_2 \), and the bottom row of all zeroes imposes the condition \( b_3 - 2b_2 + b_1 = 0 \). Hence \( Ax = b \) has a solution if and only if \( b_3 - 2b_2 + b_1 = 0 \).
(c) This is because $A$ is not full row-rank, as shown in part (b). If $AB = I$ then we could solve $Ab_1 = \text{row 1 of } I$, $Ab_2 = \text{row 2 of } I$, $Ab_3 = \text{row 3 of } I$, and every equation $Ax = b$. Actually the solution would be $x = Bb$. But in part (b) we saw that $Ax = b$ has no solution for some $b$.

(d) We perform the row reduction

$$
\begin{bmatrix}
1 & 2 & 3 & 4 & 1 \\
2 & 3 & 4 & 5 & 0 \\
3 & 4 & 5 & 6 & -1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 3 & 4 & 1 \\
0 & -1 & -2 & -3 & -2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -1 & -2 & -3 \\
0 & 1 & 2 & 3 & 2 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Then $x_p = \begin{bmatrix} -3 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution, and $s_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ and $s_4 = \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$ are special solutions forming a basis of the nullspace of $A$. Hence the general solution is

$$x = x_p + x_n = x_p + cs_3 + ds_4.$$
3 (30 pts.)

(a) Find a basis for the vector space of all real 3 by 3 symmetric matrices.

(b) Suppose $A$ is a square invertible matrix. You permute its rows by a permutation $P$ to get a new matrix $B$. How do you know that $B$ is also invertible?

(c) “If 2 matrices have the same shape and the same nullspace, then they have the same column space.” If this is true, give a reason why. If this is not true, find 2 matrices to show it’s false.

Solution

(a) The most natural basis is

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0 
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix},
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 
\end{bmatrix},
\begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 1 & 0 
\end{bmatrix}.
\]

(b) $A$ being invertible means that $A$ has full rank. Permuting the rows has no effect on the rank, so $B$ has full rank as well, and is thus invertible. (Another argument: the permutation matrix $P$ is invertible, and so $B^{-1} = (PA)^{-1} = A^{-1}P^{-1}$.)

(c) The statement is false. Example: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ have the same nullspace (the line spanned by the vector $(1, -1)$), but their column spaces differ (for $A$, it’s the line spanned by the vector $(1, 1)$, and for $B$, it’s the line spanned by the vector $(1, 2)$).
Remark (now on the web page)

The real 3 by 3 matrices form a vector space $M$. The symmetric matrices in $M$ form a subspace $S$. If you add two symmetric matrices, or multiply by real numbers, the result is still a symmetric matrix. **Problem: Find a basis for $S$.**

When I asked this question on an exam, I realized that a key point needs to be emphasized: **The basis “vectors” for $S$ must lie in the subspace.** They are 3 by 3 symmetric matrices! Then there are two requirements:

1. The basis vectors must be linearly independent.

2. Their combinations must produce every vector (matrix) in $S$.

Here is one possible basis (all symmetric) for this example:

$$
S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad S_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

$$
S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad S_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 \end{bmatrix} \quad S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 \end{bmatrix}
$$

Since this basis contains 6 vectors, the **dimension of $S$ is 6.**