Your PRINTED name is: __SOLUTIONS__

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1) T 10 2-131 K. Meszaros 2-333 3-7826 karola
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3) T 11 2-132 A. Barakat 2-172 3-4470 barakat
4) T 11 2-131 A. Osorno 2-229 3-1589 aosorno
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6) T 12 2-131 K. Meszaros 2-333 3-7826 karola
7) T 1 2-132 A. Edelman 2-343 3-7770 edelman
8) T 2 2-132 J. Burns 2-333 3-7826 burns
9) T 3 2-132 A. Osorno 2-229 3-1589 aosorno
1 (24 pts.) Suppose \( q_1, q_2, q_3 \) are orthonormal vectors in \( \mathbb{R}^3 \). Find all possible values for these 3 by 3 determinants and explain your thinking in 1 sentence each.

(a) \( \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} = \)

(b) \( \det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} = \)

(c) \( \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \times \det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix} = \)

Solution.

(a) The determinant of any square matrix with orthonormal columns (“orthogonal matrix”) is \( \pm 1 \).

(b) Here are two ways you could do this:

(1) The determinant is linear in each column:

\[
\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} = \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} \\
= \det \begin{bmatrix} q_1 & q_2 + q_3 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_3 + q_1 \end{bmatrix} \\
= \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} + \det \begin{bmatrix} q_2 & q_3 & q_1 \end{bmatrix}
\]

Both of these determinants are equal (see (c)), so the total determinant is \( \pm 2 \).
(2) You could also use row reduction. Here’s what happens:

\[
\begin{align*}
\det \begin{bmatrix} q_1 + q_2 & q_2 + q_3 & q_3 + q_1 \end{bmatrix} &= \det \begin{bmatrix} q_1 + q_2 & -q_1 + q_3 & q_3 + q_1 \end{bmatrix} \\
&= \det \begin{bmatrix} q_1 + q_2 & -q_1 + q_3 & 2q_3 \end{bmatrix} \\
&= 2 \det \begin{bmatrix} q_1 + q_2 & -q_1 + q_3 & q_3 \end{bmatrix} \\
&= 2 \det \begin{bmatrix} q_1 + q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2 \det \begin{bmatrix} q_2 & -q_1 & q_3 \end{bmatrix} \\
&= 2 \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix}
\end{align*}
\]

Again, whatever \( \det \begin{bmatrix} q_1 & q_2 & q_3 \end{bmatrix} \) is, this determinant will be twice that, or \( \pm 2 \).

(c) The second matrix is an even permutation of the columns of the first matrix (swap \( q_1/q_2 \) then swap \( q_2/q_3 \)), so it has the same determinant as the first matrix. Whether the first matrix has determinant +1 or -1, the product will be +1.
Suppose we take measurements at the 21 equally spaced times $t = -10, -9, \ldots, 9, 10$. All measurements are $b_i = 0$ except that $b_{11} = 1$ at the middle time $t = 0$.

(a) Using least squares, what are the best $\hat{C}$ and $\hat{D}$ to fit those 21 points by a straight line $C + Dt$?

(b) You are projecting the vector $b$ onto what subspace? (Give a basis.) Find a nonzero vector perpendicular to that subspace.

Solution.

(a) If the line went exactly through the 21 points, then the 21 equations

\[
\begin{bmatrix}
1 & -10 \\
1 & -9 \\
\vdots & \vdots \\
1 & 0 \\
\vdots & \vdots \\
1 & 10 \\
\end{bmatrix} \begin{bmatrix}
C \\
D \\
\end{bmatrix} = 
\begin{bmatrix}
0 \\
0 \\
\vdots \\
1 \\
\vdots \\
0 \\
\end{bmatrix}
\]

would be exactly solvable. Since we can’t solve this equation $Ax = b$ exactly, we look for a least-squares solution $A^T A \hat{x} = A^T b$.

\[
\begin{bmatrix}
21 & 0 \\
0 & 770 \\
\end{bmatrix} \begin{bmatrix}
\hat{C} \\
\hat{D} \\
\end{bmatrix} = 
\begin{bmatrix}
1 \\
0 \\
\end{bmatrix}
\]

So the line of best fit is the horizontal line $\hat{C} = \frac{1}{21}, \hat{D} = 0$.

(b) We are projecting $b$ onto the column space of $A$ above (basis: $[1 \ldots 1]^T$, $[-10 \ldots 10]^T$). There are lots of vectors perpendicular to this subspace; one is the error vector $e = b - P_A b = \frac{1}{21} \left[ (\text{ten } -1\text{'s}) \ 20 \ (\text{ten } -1\text{'s}) \right]^T$.
The Gram-Schmidt method produces orthonormal vectors $q_1, q_2, q_3$ from independent vectors $a_1, a_2, a_3$ in $\mathbb{R}^5$. Put those vectors into the columns of 5 by 3 matrices $Q$ and $A$.

(a) Give formulas using $Q$ and $A$ for the projection matrices $P_Q$ and $P_A$ onto the column spaces of $Q$ and $A$.

(b) Is $P_Q = P_A$ and why? What is $P_Q$ times $Q$? What is $\det P_Q$?

(c) Suppose $a_4$ is a new vector and $a_1, a_2, a_3, a_4$ are independent. Which of these (if any) is the new Gram-Schmidt vector $q_4$? ($P_A$ and $P_Q$ from above)

1. $\frac{P_Qa_4}{\|P_Qa_4\|}$
2. $\frac{a_4 - \frac{a_1^T}{a_1^T}a_1 - \frac{a_2^T}{a_2^T}a_2 - \frac{a_3^T}{a_3^T}a_3}{\|\text{norm of that vector}\|}$
3. $\frac{a_4 - P_Aa_4}{\|a_4 - P_Aa_4\|}$

Solution.

(a) $P_A = A(A^TA)^{-1}A^T$ and $P_Q = Q(Q^TQ)^{-1}Q^T = QQ^T$.

(b) $P_A = P_Q$ because both projections project onto the same subspace. (Some people did this the hard way, by substituting $A = QR$ into the projection formula and simplifying. That also works.) The determinant is zero, because $P_Q$ is singular (like all non-identity projections): all vectors orthogonal to the column space of $Q$ are projected to 0.

(c) Answer: choice 3. (Choice 2 is tempting, and would be correct if the $a_i$ were replaced by the $q_i$. But the $a_i$ are not orthogonal!)
4 (22 pts.) Suppose a 4 by 4 matrix has the same entry \( \times \) throughout its first row and column. The other 9 numbers could be anything like 1, 5, 7, 2, 3, 99, \( \pi \), e, 4.

\[
A = \begin{bmatrix}
\times & \times & \times & \times \\
\times & \text{any numbers} & & \\
\times & \text{any numbers} & & \\
\times & \text{any numbers} & & \\
\end{bmatrix}
\]

(a) The determinant of \( A \) is a polynomial in \( \times \). What is the largest possible degree of that polynomial? Explain your answer.

(b) If those 9 numbers give the identity matrix \( I \), what is \( \det A \)? Which values of \( \times \) give \( \det A = 0 \)?

\[
A = \begin{bmatrix}
\times & \times & \times & \times \\
\times & 1 & 0 & 0 \\
\times & 0 & 1 & 0 \\
\times & 0 & 0 & 1 \\
\end{bmatrix}
\]

Solution.

(a) Every term in the big formula for \( \det(A) \) takes one entry from each row and column, so we can choose at most two \( \times \)'s and the determinant has degree 2.

(b) You can find this by cofactor expansion; here's another way:

\[
\det(A) = \times \det \begin{bmatrix} 1 & \times & \times & \times \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \times \det \begin{bmatrix} 1 - 3\times & \times & \times & \times \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \times (1 - 3\times) \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

This is zero when \( \times = 0 \) or \( \times = \frac{1}{3} \).