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1 (30 pts.)

Let \( A = \begin{bmatrix} 0 & 0 \\ 6 & 9 \\ 2 & 3 \end{bmatrix} \).

(a) (6 pts.) Circle the best answer: The column space of \( A \) is a
   
   a. point  b. line  c. plane  d. three dimensional space. Explain very briefly.
   
   The matrix has rank 1, seen by inspection or by elimination.

(b) (6 pts.) True or False: The row space of \( A \) is a vector subspace of \( R^3 \), i.e., consists of a collection of vectors with three components that are closed under all linear combinations. Explain very briefly.

   The rowspace is a subspace of \( R^2 \), collections of vectors with two components.

(c) (6 pts.) Circle the best answer:
   
   a. Matrix \( A \) has full column rank.
   b. Matrix \( A \) has full row rank.
   c. Matrix \( A \) has neither full column rank nor full row rank.

   Explain very briefly.

   The rank is both smaller than than the number of rows and the number of columns.
(d) (12 pts.) Let \( b = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \). Find the complete solution to \( Ax = b \).

The second column of \( A \) is free.

The particular solution is then \( x_p = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} \). There is one special solution \( \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \).

The complete solution consists of all vectors of the form

\[
\begin{bmatrix} 1/2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}.
\]
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2 (23 pts.)

A is a square $3 \times 3$ matrix whose LU decomposition exists with no row exchanges. Carefully provide a count of the exact number of operations required to compute the three parameters $l_{12}, l_{13}$ and $l_{23}$ of L and the six parameters of U. The questions below count first all the divisions, then all the multiplications, and then all the subtractions that occur. Avoid any unnecessary operations. (Operations on the elements being eliminated are unnecessary since that element’s ultimate fate is known to be 0.)

| Note: Other answers maybe ok with explanation. |

(a) (5 pts.) We recall that computation of each multiplier $l_{ij}$ requires one division. The exact number of divisions in the LU decomposition of our $3 \times 3$ A is 3.

Each of the three multipliers requires one division to compute.

(b) (9 pts.) We recall that multipliers $l_{ij}$ multiply the $j$th row but only to the right of column $i$. The exact number of multiplications in the entire LU decomposition of our $3 \times 3$ A is

\[5 = 2^2 + 1^2\]

The first row(column) is the pivot row(column) for the first two eliminations. First $l_{21}$ scales $A_{12}$ and $A_{13}$ as a byproduct of the elimination of the (2,1) entry. Then $l_{31}$ also scales $A_{12}$ and $A_{13}$. By the time we have computed $l_{32}$, it multiplies the value now in $A_{23}$.

In general, we phrased the question in a way that counted for the fact that no computation need occur in the first row or first column. The total number of multiplications in general would then be $(n - 1)^2 + \ldots + 1.
(c) (9 pts.) We recall that after \( l_{ij} \) does its job of multiplying row \( j \) to the right of column \( i \), we subtract row \( j \) from row \( i \) but only to the right of column \( i \). The exact number of subtractions in the entire LU decomposition of our \( 3 \times 3 \) \( A \) is \( \boxed{5} \).

\[
5 = 2^2 + 1. \text{ The subtraction count is always the same as the multiplication count.}
\]
3 (27 pts.)

$A$ is a matrix which has two special solutions to $Ax = 0$. All other solutions, we recall, are linear combinations of the two special solutions. The two special solutions are

\[
\begin{bmatrix}
3 \\
1 \\
4 \\
0 \\
5
\end{bmatrix}
\quad \text{and} \quad
\begin{bmatrix}
2 \\
0 \\
2 \\
1 \\
2
\end{bmatrix}.
\]

(a) (9 pts.) What is $r =$rank$(A)$? What is the dimension of the column space $C(A)$? What is the dimension of the nullspace $N(A)$? Very briefly explain your three numbers.

<table>
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<th>$r = 3 =$ (5 columns)-(2 special solutions)</th>
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<tbody>
<tr>
<td>dim$(C(A))=r = 3$</td>
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<tr>
<td>dim$(N(A))=2$=number of special solutions</td>
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(b) (9 pts.) $B$ is a matrix that is the same as $A$ except that its second row is 
(row 2 of $A$)-(row 1 of $A$). What is a basis for the nullspace $N(B)$?

The two special solutions above, since $N(B) = N(A)$.

(c) (9 pts.) $C$ is a matrix that is the same as $A$ except that its second column is 
(column 2 of $A$)-(column 1 of $A$). What is a basis for the nullspace $N(C)$? (Hint: If $M$
is invertible, it may be useful to know that if $y$ is in $N(C)$, then $M^{-1}y$ is in $N(CM)$.)

$C = AM$, where $M$ is the matrix given in the hint. We recognize matrices such as

$M$. The inverse of $M$ looks the same as $M$ except the $(2,1)$ entry is $+1$ not $-1$.

$M^{-1}$ takes a vector and replaces the first component with the sum of the first two

components. Thus the new basis consists of the multiplication of $M^{-1}$ times the

previous basis vectors:

$\begin{bmatrix}
4 \\
1 \\
4 \\
0 \\
5
\end{bmatrix}$ and

$\begin{bmatrix}
2 \\
0 \\
2 \\
1 \\
2
\end{bmatrix}$.
4  (20 pts.)

The next question concerns $M_4$, the 16 dimensional space of $4 \times 4$ real matrices.

(a) (10 pts.) True or False. The twenty-four $4 \times 4$ permutation matrices are independent members of $M_4$? Explain briefly.

At most 16 members of this space can be independent because it is a 16 dimensional vector

(b) (10 pts.) True or False. The twenty-four $4 \times 4$ permutation matrices span $M_4$? (Hint: is any row sum possible?) Explain briefly.

To span the space, every matrix must be a linear combination of permutation matrices. This is impossible. Why? Because every row of a permutation matrix sums to 1, so every row of a linear combination of permutation matrices is the same. Try it! Thus a matrix with different row sums can not be a linear combination of permutation matrices.
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