Your PRINTED name is:________________________

Please circle your recitation:

1  T 9  2-132  Kestutis Cesna\ vicius  2-089  2-1195  kestutis
2  T 10  2-132  Niels Moeller  2-588  3-4110  moller
3  T 10  2-146  Kestutis Cesna\ vicius  2-089  2-1195  kestutis
4  T 11  2-132  Niels Moeller  2-588  3-4110  moller
5  T 12  2-132  Yan Zhang  2-487  3-4083  yanzhang
6  T 1  2-132  Taedong Yun  2-342  3-7578  tedyun
Let \( A = \begin{pmatrix} .5 & 0 & 0 \\ .5 & .9 & 0 \\ 0 & .1 & 1 \end{pmatrix} \).

1. (4 pts) True or False: The matrix \( A \) is Markov.

   True. Markov matrices have columns that sum to 1 and have non-negative entries. The answer of false applies to what is known as “Positive Markov Matrices.”

2. (6 pts) Find a vector \( x \neq 0 \) and a scalar \( \lambda \) such that \( A^T x = \lambda x \).

   The obvious choice is \((1,1,1)\) with \( \lambda = 1 \) as this is the column sum property. Also easy to see is \((1,0,0)\) with \( \lambda = 0.5 \).
3. (4 pts) True or False: The matrix $A$ is diagonalizable. (Explain briefly.)

True. The three eigenvalues, on the diagonal, are distinct.

4. (4 pts) True or False: One singular value of $A$ is $\sigma = 0$. (Explain briefly.)

False. The matrix is nonsingular, since it has no zero eigenvalues. Nonsingular square matrices have all $n$ singular values positive.

5. (6 pts) Find the three diagonal entries of $e^{At}$ as functions of $t$.

They are $e^t$, $e^{0.5t}$, $e^{0.9t}$.
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2. (30 pts.)

1. (5 pts) An orthogonal matrix $Q$ satisfies $Q^T Q = QQ^T = I$. What are the $n$ singular values of $Q$?

They are all 1. The singular values are the positive square roots of the eigenvalues of $QQ^T = Q^T Q = I$.

2. (10 pts) Let $A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix}$. Find an SVD, meaning $A = U \Sigma V^T$, where $U$ and $V$ are orthogonal, and $\Sigma = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{pmatrix}$ is diagonal with $\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq 0$. (Be sure that the factorization is correct and satisfies all stated requirements.)

$A = \begin{pmatrix} 1 & -2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. The singular values are in decreasing order and are positive. One can compute $AA^T$ and $A^TA$, but easier to rig the permutation matrices and correct the sign.
3. (15 pts) The $2 \times 2$ matrix $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$, where $\sigma_1 > \sigma_2 > 0$ and both $u_1, u_2$ and $v_1, v_2$ are orthonormal bases for $\mathbb{R}^2$.

The set of all vectors $x$ with $\|x\| = 1$ describes a circle in the plane. What shape best describes the set of all vectors $Ax$ with $\|x\| = 1$? Draw a general picture of that set labeling all the relevant quantities $\sigma_1, \sigma_2, u_1, u_2$ and $v_1, v_2$. (Hint: Why are $u_1, u_2$ relevant and $v_1, v_2$ not relevant?)

The SVD rotates (or reflects) the circle with $V^T$, scales to an ellipse with axes in the coordinate directions through $\Sigma$, and then a rotated ellipse with axes in the direction $u_1$ and $u_2$ after $U$ is applied. The $\Sigma$ scales the $x$ and $y$ axes by $\sigma_1$ and $\sigma_2$ respectively, and $\sigma_1$ is the longer of the two.
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3 (16 pts.)

1. (6 pts) Let \( x \neq 0 \) be a vector in \( \mathbb{R}^3 \). How many eigenvalues of \( A = xx^T \) are positive? zero? negative? (Explain your answer.) (Hint: What is the rank?)

\( A \) is symmetric positive semidefinite and rank 1, so there are 1 positive, 2 zero, and no negative eigenvalues.

2. (6 pts) a) What are the possible eigenvalues of a projection matrix?

0 and 1 (Since \( P^2 = P, \lambda^2 = \lambda \)).

b) True or False: every projection matrix is diagonalizable.

True, every projection matrix is symmetric, hence diagonalizable.

3. (4 pts) True or False: If every eigenvalue of \( A \) is 0, then \( A \) is similar to the zero matrix.

False. A Jordan block with zero eigenvalues is not similar to the zero matrix for \( n > 1 \).
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4 \hspace{1em} (30 \text{ pts.})

Consider the matrix \( A = \begin{pmatrix} \mathbf{x} & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) with parameter \( x \) in the (1,1) position.

1. (10 pts) Specify all numbers \( x \), if any, for which \( A \) is positive definite. (Explain briefly.)

No \( x \), the matrix is clearly singular with two equal rows and two equal columns.

2. (10 pts) Specify all numbers \( x \), if any, for which \( e^A \) is positive definite. (Explain briefly.)

The eigenvalues of \( e^A \) are the exponentials of the eigenvalues of the matrix \( A \). Since \( A \) is symmetric the eigenvalues are real, and thus exponentials are positive. A symmetric matrix with positive eigenvalues is positive definite.
3. (10 pts) Find an $x$, if any, for which $4I - A$ is positive definite. ( Explain briefly.)

One can take any $x < 3$. The easiest choice is $x = 1$. With this guess the matrix has two eigenvalues 0 and one eigenvalue 3 both less than 4, so $4 - \lambda > 0$ for all three eigenvalues. Systematically, one can consider the three upper left determinants of $4I - A$ which are $4 - x$, $11 - 3x$, and $24 - 8x$. They are all positive if and only if $x < 3$. 
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