1. Do problem 4 from 4.4.

**Solution.** (a) The matrix \( Q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) has orthonormal columns but

\[
QQ^T = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.
\]

(b) The vectors \([0], [0] \) in \( \mathbb{R} \) are orthogonal but are not linearly independent.

(c) I claim that

\[
\begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix}
\]

is such. These three vectors are clearly orthonormal. Therefore they are linearly independent (every set of pairwise orthogonal nonzero vectors is linearly independent - check this!). But any three linearly independent vectors in \( \mathbb{R}^3 \) form a basis and this verifies my claim.

2. Do problem 19 from 4.4.

**Solution.** If \( A = QR \) then \( A^TA = R^TR = \begin{bmatrix} \text{lower} \end{bmatrix} \) triangular times \( \begin{bmatrix} \text{upper} \end{bmatrix} \) triangular.

Let \( c_1, c_2 \) denote the columns of \( A \). Gram-Schmidt gives

\[
q'_1 = c_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \quad q'_2 = c_2 - \frac{\langle q'_1, c_2 \rangle}{\langle q'_1, q'_1 \rangle} q'_1 = \begin{bmatrix} 1 \\ -9/9 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.
\]

Scaling to get unit lengths gives

\[
q_1 = \frac{q'_1}{\|q'_1\|} = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \quad q_2 = \frac{q'_2}{\|q'_2\|} = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix}.
\]

Since

\[
R = \begin{bmatrix} \langle c_1, q_1 \rangle & \langle c_2, q_1 \rangle \\ 0 & \langle c_2, q_2 \rangle \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}
\]

the desired \( A = QR \) decomposition reads

\[
\begin{bmatrix} -1 & 1 \\ 2 & 1 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \\ 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 3 & 3 \\ 0 & 3 \end{bmatrix}.
\]

3. Do problem 37 from 4.4. Hint: Find a vector in \( c(A) \) that is orthogonal to \( c(Q) \), then normalize.
Solution. The projection of \( a \) onto the column space of \( Q \) is \( Pa = QQ^T a \). So if you subtract \( \frac{QQ^T a}{\|a - QQ^T a\|} \) and divide by \( \|a - QQ^T a\| \) you will get the new orthogonal vector \( q = \frac{a - QQ^T a}{\|a - QQ^T a\|} \). This is of unit length and to check that \( q \) is orthogonal to the column space of \( Q \) we simply show that the projection of \( q \) onto \( C(Q) \) is zero:

\[
P_q = \frac{P(a - QQ^T a)}{\|a - QQ^T a\|} = \frac{QQ^T(a - QQ^T a)}{\|a - QQ^T a\|} = \frac{(QQ^T a - Q(Q^T Q)Q^T a)}{\|a - QQ^T a\|} = 0.
\]

4. Do problem 2 from 8.5.

Solution. To show that the corresponding functions are orthogonal we simply need to show that appropriate integrals vanish:

\[
\int_{-1}^{1} 1 \cdot x \, dx = \frac{x^2}{2}\bigg|_{-1}^{1} = 0,
\]
\[
\int_{-1}^{1} 1 \cdot \left( x^2 - \frac{1}{3} \right) \, dx = \left( \frac{x^3}{3} - \frac{x}{3} \right)\bigg|_{-1}^{1} = 0,
\]
\[
\int_{-1}^{1} x \cdot \left( x^2 - \frac{1}{3} \right) \, dx = \left( \frac{x^4}{4} - \frac{x^2}{6} \right)\bigg|_{-1}^{1} = 1 - \frac{1}{6} - \left( \frac{1}{4} - \frac{1}{6} \right) = 0.
\]

Writing \( f(x) = 2x^2 \) as a combination of those functions simply amounts to

\[
f(x) = 2x^2 = 2 \left( x^2 - \frac{1}{3} \right) + \frac{2}{3} \cdot 1.
\]

5. Do problem 4 from 8.5.

Solution. Note that \( x^3 - cx \) is perpendicular to 1 regardless of \( c \):

\[
\int_{-1}^{1} 1 \cdot (x^3 - cx) \, dx = \frac{x^4}{4} - \frac{cx^2}{2}\bigg|_{-1}^{1} = 0.
\]

For \( x^3 - cx \) to be perpendicular to \( x \) we must have

\[
\int_{-1}^{1} x \cdot (x^3 - cx) \, dx = \left( \frac{x^5}{5} - \frac{cx^3}{3} \right)\bigg|_{-1}^{1} = \frac{1}{5} - \frac{c}{3} - \left( -\frac{1}{5} - \frac{c}{3} \right) = 0,
\]

i.e., \( c = \frac{3}{5} \). It remains to show that with this \( c \) the function \( x^3 - cx \) is also perpendicular to \( x^2 - \frac{1}{3} \):

\[
\int_{-1}^{1} \left( x^2 - \frac{1}{3} \right) \left( x^3 - \frac{3x}{5} \right) \, dx = \int_{-1}^{1} \left( x^5 - \frac{14}{15} x^3 + \frac{x}{5} \right) \, dx
\]
\[
= \left( \frac{x^6}{6} - \frac{14}{15} \frac{x^4}{4} + \frac{x^2}{10} \right)\bigg|_{-1}^{1} = 0,
\]

where to obtain the last equality we have observed that the function in parentheses is even.
6. Do problem 12 from 8.5.

Solution. The 5 by 5 “differentiation matrix” is
\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & -2 & 0
\end{bmatrix}
\]
which succintly expresses the information about expressing the derivatives of the five functions in terms of those same functions:
\[
\begin{align*}
1' &= 0, \\
(\cos x)' &= -\sin x, \\
(\sin x)' &= \cos x, \\
(\cos 2x)' &= -2\sin 2x, \\
(\sin 2x)' &= 2\cos 2x.
\end{align*}
\]

7. (This problem is worth 20 points) In MATLAB or your favorite language, create 2n-length discrete versions of \( q_1 = 1/\sqrt{n}\cos(x) \) and \( q_2 = 1/\sqrt{n}\cos(3x) \) by taking equal sized samples from \( 0 \) to \( 2\pi \), taking care to include \( 0 \) but exclude \( 2\pi \). This means we want to think of each of these as column vectors \( [x_0, \ldots, x_{2n-1}]^T \) where \( x_i = i\pi/n \).

In MATLAB this is \( x=(0:(2*n-1))'*pi/n \). (before you go on, test to yourself that they’re unit vectors). Let \( Q = [q_1 q_2] \).

(a) Derive an identity for \( \cos(3x) \) in terms of \( \cos(x) \) (hint: you can use sum to product formulae). Use this identity to prove that \( \cos(x)^3 \) is in the span of \( \cos(x) \) and \( \cos(3x) \).

Solution. Sure we can use sum to product formulae to express \( \cos 3x = \cos(2x + x) \) in terms of trigonometric functions of arguments \( x \) and \( 2x \) and then use double angle formulas to get rid of all \( \cos 2x \) and \( \sin 2x \). But we can also use complex numbers to derive the identity in a much slicker way! Observe the Euler identity
\[
e^{ix} = \cos x + i\sin x
\]
and then cube it. You will get
\[
\cos 3x + i\sin 3x = e^{3ix} = (\cos x + i\sin x)^3 = (\cos x)^3 - 3\cos x(\sin x)^2 + i(3\cos x^2 \sin x - (\sin x)^3),
\]
and consequently
\[
\cos 3x = (\cos x)^3 - 3\cos x(\sin x)^2 = (\cos x)^3 - 3\cos x(1 - (\cos x)^2) = 4(\cos x)^3 - 3\cos x.
\]
It is immediately clear that \( \cos(x)^3 \) lies in the span of \( \cos 3x \) and \( \cos x \) because
\[
(\cos x)^3 = \frac{1}{4}\cos 3x + \frac{3}{4}\cos x.
\]
(b) Project \( b = \cos(x)^3 \) into the column space of \( Q \) as to obtain the best least squares fit (for a shortcut, see blue line under eq. 4 on page 233). This should give some expansion. Does \( b \) equal its projection? What does this have to do with the previous part of the problem (there should really only be one reasonable interpretation of this question)?

**Solution.** [See MATLAB code]

(c) Now project \( b = \cos(x)^5 \) onto the column space of \( Q \). Does \( b \) equal its projection? If the answer is different from the previous part, why not?

**Solution.** [See MATLAB code]

8. Do problem 14 from 5.1.

**Solution.** As required, we do row operations:

\[
\begin{vmatrix}
1 & 2 & 3 & 0 \\
2 & 6 & 6 & 1 \\
-1 & 0 & 0 & 3 \\
0 & 2 & 0 & 7 \\
\end{vmatrix}
= \det
\begin{vmatrix}
1 & 2 & 3 & 0 \\
0 & 2 & 0 & 1 \\
0 & 2 & 3 & 3 \\
0 & 2 & 0 & 7 \\
\end{vmatrix}
= \det
\begin{vmatrix}
1 & 2 & 3 & 0 \\
0 & 2 & 0 & 1 \\
0 & 0 & 3 & 2 \\
0 & 0 & 0 & 6 \\
\end{vmatrix}
= 1 \cdot 2 \cdot 3 \cdot 6 = 36.
\]

Similarly,

\[
\begin{vmatrix}
2 & -1 & 0 & 0 \\
-1 & 2 & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{vmatrix}
= \det
\begin{vmatrix}
2 & -1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & 0 \\
0 & -1 & 2 & -1 \\
0 & 0 & -1 & 2 \\
\end{vmatrix}
= \det
\begin{vmatrix}
2 & -1 & 0 & 0 \\
0 & \frac{3}{2} & -1 & 0 \\
0 & 0 & \frac{4}{3} & -1 \\
0 & 0 & 0 & 5 \times \frac{1}{4} \\
\end{vmatrix}
= 2 \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} = 5.
\]


**Solution.** Even though projection matrices \( P = A(A^T A)^{-1} A^T \) are square, \( A \) appearing in the formula need not be. Therefore, it does not make sense to talk of \( \det A \) and the “proof” breaks down.

MATLAB code

```matlab
%%%%%%%%%%%%%%%%%%%
%Problem 7 (b),(c)%
%%%%%%%%%%%%%%%%%%%

n=10;
```
\[ x = (0: (2n-1)) \pi / n; \]
\[ q1 = \cos(x) / \sqrt{n}; \]
\[ q2 = \cos(3x) / \sqrt{n}; \]
\[ \text{norm}(q1) \]
\[ \text{ans} = 1 \]
\[ \text{norm}(q2) \]
\[ \text{ans} = 1 \]
\[ Q = [q1 \ q2]; \]
\[ b = (\cos(x).^3) / \sqrt{n}; \]
\% this shows projecting the \cos^3 \ vector get
\% itself back, since the norm of the difference
\% is basically 0. You can also just
\% display the two separately and look by eye.
\[ \text{norm}((Q*(Q'*Q)^(-1)*Q'*b)-b) \]
\[ \text{ans} = 1.6059e-16 \]
\% now we do the \cos^5 \ vector. Here the difference
\% is very far from 0
\[ c = (\cos(x).^5) / \sqrt{n}; \]
\[ \text{norm}((Q*(Q'*Q)^(-1)*Q'*c)-c) \]
\[ \text{ans} = 0.0625 \]
\% an additional thing you can do to check the
\% coefficients:
\% \cos(3x) = 4*(\cos(x))^3 - 3*\cos(x),
\% hence \cos(x)^3 = \frac{1}{4}\cos(3x) + \frac{3}{4}\cos(x)
\% the following shows this:

\texttt{Q'*b}

\texttt{ans =}

\begin{verbatim}
  0.7500
  0.2500
\end{verbatim}

diary off