Exam Solutions

Problem 1

(a) Do Gram-Schmidt orthogonalization for the vectors \( a_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, a_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, a_3 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \).

(b) Find the \( A = QR \) decomposition for the matrix \( \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \).

(c) Find the projection of the vector \((1, 0, 0)^T\) onto the line spanned by the vector \((1, 1, 1)^T\).

(d) Find the projection of the vector \((1, -1, 0)^T\) onto the plane \( x + y + z = 0 \) in \( \mathbb{R}^3 \).

(e) Find the least squares solution \( \hat{x} \) for the system

\[
\begin{pmatrix}
1 & -1 & 1 & 0 \\
1 & 0 & 1 & 2 \\
1 & 1 & 0 & 1 \\
1 & 2 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z \\
w
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
10 \\
0
\end{pmatrix}.
\]

Solutions:

(a) \( a_1 \) and \( a_2 \) are already orthogonal so \( b_1 = a_1 \) and \( b_2 = a_2 \).

\[
b_3 = a_3 - \frac{a_3 \cdot b_1}{b_1 \cdot b_1} b_1 - \frac{a_3 \cdot b_2}{b_2 \cdot b_2} b_2 = a_3 - 2a_1 - 2a_2 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.
\]

(b) Gram-Schmidt orthogonalization on \( a_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \) and \( a_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \) gives \( b_1 = a_1 \) and \( b_2 = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \)

so \( Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Inspection gives \( R = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \).

\[
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 \\
1 & 0 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1 \\
1
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
1 \\
1
\end{pmatrix} / 3.
\]

(d) The vector already lies in the plane so projection does nothing: \( (1, -1, 0)^T \).

(e) We must solve

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
-1 & 0 & 1 & 2
\end{pmatrix}
\begin{pmatrix}
1 & -1 \\
1 & 0 \\
1 & 1 \\
1 & 2
\end{pmatrix}
\hat{x} = 
\begin{pmatrix}
0 \\
0 \\
10 \\
0
\end{pmatrix}, \text{ i.e. } 
\begin{pmatrix}
4 & 2 \\
2 & 6
\end{pmatrix}
\hat{x} = 
\begin{pmatrix}
10 \\
10
\end{pmatrix}.
\]

So \( \hat{x} = \begin{pmatrix} 4 \\ 2 \\ 6 \end{pmatrix}^{-1} \begin{pmatrix} 10 \\ 10 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \).
Problem 2
Let \( A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \).

(a) Calculate \( \det(A) \).

(b) Explain why \( A \) is an invertible matrix. Find the \((2,3)\) entry of the inverse matrix \( A^{-1} \).

(c) Notice that all sums of entries in rows of \( A \) are the same. Explain why this implies that \((1,1,1)^T\) is an eigenvector of \( A \). What is the corresponding eigenvalue \( \lambda_1 \).

(d) Find two other eigenvalues \( \lambda_2 \) and \( \lambda_3 \) of \( A \).

(e) Find the projection matrix \( P \) for the projection onto the column space of \( A \).

Solutions:

(a) Using row operations we see that \( \det(A) = \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} \). Moreover, using the cofactor formula, \( \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & -1 \\ 0 & -1 & -3 \end{pmatrix} = \det \begin{pmatrix} 1 & -1 \\ -1 & -3 \end{pmatrix} = -3 - 1 = -4 \).

(b) \( \det(A) = -4 \neq 0 \). Matrices with non-zero determinants are invertible. The \((2,3)\) entry of \( A^{-1} \) is given by
\[
\frac{C_{3,2}}{\det A} = \frac{1}{4} \det \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = -\frac{1}{4}.
\]

(c) \( A(1,1,1)^T = 4(1,1,1)^T \) shows directly that \((1,1,1)^T\) is an eigenvector for \( A \) with eigenvalue \( \lambda_1 = 4 \).

(d) We have \( \lambda_1 + \lambda_2 + \lambda_3 = \text{tr}(A) = 4 \) and \( \lambda_1\lambda_2\lambda_3 = \det(A) = -4 \). Remembering that \( \lambda_1 = 4 \) this gives \( \lambda_2 + \lambda_3 = 0 \) and \( \lambda_2\lambda_3 = -1 \). Up to reordering, this system of equations has a unique solution, \( \lambda_2 = 1, \lambda_3 = -1 \).

(e) Since \( \det(A) \neq 0 \), \( A \) is invertible and so the column space of \( A \) is all of \( \mathbb{R}^3 \). The projection matrix onto \( \mathbb{R}^3 \) is the identity \( I \).
Problem 3

(a) Calculate the area of a triangle on the plane $\mathbb{R}^2$ with the vertices $(1, 0), (0, 1), (3, 3)$ using the determinant.

(b) Find all values of $x$ for which the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$ has an eigenvalue equal to 2.

(c) Diagonalize the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(d) Calculate the power $B^{2014}$ of the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(e) Let $Q$ be any matrix which is symmetric and orthogonal. Find $Q^{2014}$. Explain your answer.

Solutions:

(a) Translation by $(-1, 0)$ is an isometry and so it is equivalent to find the area of a triangle with the vertices $(0, 0), (-1, 1), (2, 3)$. This is given by

$$\frac{1}{2} \left| \det \begin{pmatrix} -1 & 1 \\ 2 & 3 \end{pmatrix} \right| = \frac{5}{2}.$$ 

(b) $A$ has an eigenvalue equal to 2 if and only if the matrix $A - 2I$ is singular. Thus, $A$ has an eigenvalue equal to 2 if and only if $\det(A - 2I) = 0$. But

$$\det(A - 2I) = \det \begin{pmatrix} -1 & x \\ 1 & -1 \end{pmatrix} = 1 - x.$$ 

So $\det(A - 2I) = 0$ if and only if $1 - x = 0$, i.e. $x = 1$.

(c) Since $B$ is diagonal its eigenvalues can be read off from the diagonal $\lambda_1 = 1$ and $\lambda_2 = -1$. We find corresponding eigenvectors $(1, 0)^T$ and $(1, -1)^T$. So $B = SAS^{-1}$, where

$$\Lambda = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } S = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}.$$ 

By chance we have $S = S^{-1}$.

(d) $B^2 = SA^2S^{-1} = SIS^{-1} = I$, so $B^{2014} = (B^2)^{1007} = I$.

(e) Since $Q$ is orthogonal we have $Q^TQ = I$. Since $Q$ is symmetric we have $Q^T = Q$. Thus

$$Q^2 = QQ = Q^TQ = I$$

and $Q^{2014} = (Q^2)^{1007} = I$. 

3
Problem 4

Consider the Markov matrix

\[
A = \begin{pmatrix}
0 & 1/3 & 1/3 & 0 \\
1/2 & 0 & 1/3 & 1/2 \\
1/2 & 1/3 & 0 & 1/2 \\
0 & 1/3 & 1/3 & 0
\end{pmatrix}.
\]

(a) Three of the eigenvalues are 1, 0, \(-1/3\). Find the fourth eigenvalue of \(A\).

(b) Find the determinant \(\det(A)\).

(c) Find the eigenvector of the transposed matrix \(A^T\) with eigenvalue \(\lambda_1 = 1\).

(d) Find the eigenvector of the matrix \(A\) with the eigenvalue \(\lambda_1 = 1\). (Hint: notice that the nonzero entries in each column of \(A\) are the same.)

(e) Find the limit of \(A^k(1,0,0,0)^T\) as \(k \to +\infty\).

Solutions:

(a) Since \(\text{tr}(A) = 0\) the sum of the eigenvalues are 0. Thus, the fourth eigenvalue must be \(-2/3\).

(b) The determinant is the product of the eigenvalues, which is 0.

(c) \((1,1,1,1)A = (1,1,1,1)\) and so the eigenvector of \(A^T\) with eigenvalue \(\lambda_1 = 1\) is \((1,1,1,1)^T\).

(d) The Markov matrix \(A\) corresponds to a random walk on the graph with four nodes 1, 2, 3, 4 connected by the edges (1,2), (1,3), (2,3), (2,4), (3,4). The degrees of the nodes are 2, 3, 3, 2. Thus the vector \((2,3,3,2)^T\) is an eigenvector with eigenvalue \(\lambda_1 = 1\).

(e) Let \(v_1 = (2,3,3,2)^T\) and let \(v_2, v_3\) and \(v_4\) be eigenvectors for 0, \(-1/3\), \(-2/3\), respectively. Then there exist \(c_1, \ldots, c_4 \in \mathbb{R}\) with

\[
(1,0,0,0)^T = c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4.
\]

Thus

\[
A^k(1,0,0,0)^T = c_1v_1 + \frac{(-1)^kc_3}{3^k}v_3 + \frac{(-2)^kc_4}{3^k}v_4 \to c_1v_1, \text{ as } k \to +\infty
\]

To find \(c_1\) we recall that \((1,1,1,1)A = (1,1,1,1)\). By induction we obtain

\[
(1,1,1,1)A^k = (1,1,1,1)
\]

and so \((1,1,1,1)A^k(1,0,0,0)^T = (1,1,1,1)(1,0,0,0)^T = 1\). Letting \(k \to +\infty\) we obtain

\[
(1,1,1,1)c_1v_1 = 1
\]

so that \(c_1 = 1/(1,1,1,1)v_1) = 1/10\). The answer to the question is \((2,3,3,2)^T/10\).