Recitation 10

The matrix of a linear transformation

**Definition 0.1.** A *vector space* is a set \( V \) together with operations \( + : V \times V \to V \) and \( \cdot : \mathbb{R} \times V \to V \) called “addition” and “scalar multiplication,” respectively, satisfying various axioms, which should be intuitive by now.

**Definition 0.2.** We say that vectors \( v_1, \ldots, v_n \in V \) are *linearly dependent* if there exists \( c_1, \ldots, c_n \in \mathbb{R} \) not all zero, with \( c_1 v_1 + \ldots + c_n v_n = 0 \). If \( v_1, \ldots, v_n \) are not linearly dependent then we say they are *linearly independent*.

**Definition 0.3.** \( v_1, \ldots, v_n \in V \) are said to be a *basis* for \( V \) if \( v_1, \ldots, v_n \) are linearly independent and any \( v \in V \) can be written as a linear combination of \( v_1, \ldots, v_n \): that is \( v = c_1 v_1 + \ldots + c_n v_n \) for some \( c_1, \ldots, c_n \in \mathbb{R} \).

**Definition 0.4.** If \( V \) and \( W \) are vector spaces, a *linear transformation* \( T : V \to W \) is a function such that \( T(v + cw) = T(v) + cT(w) \) for any \( v, w \in V \) and \( c \in \mathbb{R} \).

**Definition 0.5.** Suppose \( V \) and \( W \) are vector spaces with bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \), respectively. Then for each \( j \in \{1, \ldots, n\} \), \( T(v_j) \) is a linear combination of the \( w_1, \ldots, w_m \). So there exist

\[
a_{1,j}, \ldots, a_{m,j} \text{ with } T v_j = a_{1,j} w_1 + a_{2,j} w_2 + \ldots + a_{m,j} w_m.
\]

The *matrix of \( T \)* with respect to the bases \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) is the \( m \times n \) matrix \( A \), with entries \( a_{i,j} \).

**The most basic example of the matrix of a linear transformation**

1. \( \mathbb{R}^n \) is a vector space:

\[
(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n) \text{ and } c(x_1, \ldots, x_n) = (cx_1, \ldots, cx_n).
\]

2. Let \( e_i \in \mathbb{R}^n \) be the vector with 1 in the \( i \)th entry and 0 in the other entries. Then \( e_1, \ldots, e_n \) are linearly independent in \( \mathbb{R}^n \).

3. \( e_1, \ldots, e_n \) is a basis for \( \mathbb{R}^n \).

4. An \( m \times n \) matrix \( A \) defines a linear transformation \( T : \mathbb{R}^n \to \mathbb{R}^m \) by \( T(x) = Ax \).

5. \( \mathbb{R}^n \) has basis \( e_1, \ldots, e_n \) and \( \mathbb{R}^m \) has basis \( e'_1, \ldots, e'_m \), where we use the prime to highlight the vectors are in \( \mathbb{R}^m \) as opposed to \( \mathbb{R}^n \), but they have an identical definition. We have

\[
A e_j = a_{1,j} e'_1 + a_{2,j} e'_2 + \ldots + a_{m,j} e'_m
\]

and so the matrix of the linear transformation \( T(x) = Ax \) with respect to the bases, \( e_1, \ldots, e_n \) and \( e'_1, \ldots, e'_m \) is \( A \).
Changing basis

**Definition 0.6.** If $V$ is a vector space and $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$ are two bases for $V$. Then there exists an invertible $n \times n$ matrix $B$ with entries $b_{i,j}$ such that

$$v'_j = b_{1,j}v_1 + b_{2,j}v_2 + \ldots + b_{n,j}v_n.$$

$B$ is said to be the *basis change matrix* from $v_1, \ldots, v_n$ to $v'_1, \ldots, v'_n$.

Suppose $V$ and $W$ are vector spaces. Suppose that $V$ has bases $v_1, \ldots, v_n$ and $v'_1, \ldots, v'_n$, that $W$ has bases $w_1, \ldots, w_m$ and $w'_1, \ldots, w'_m$, and that $T : V \to W$ is a linear transformation.

If $T$ has matrix $A$ with respect to the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$, then $T$ has matrix $C^{-1}AB$ with respect to the bases $v'_1, \ldots, v'_n$ and $w'_1, \ldots, w'_m$, where $B$ and $C$ denote the basis change matrices from the unprimed bases to the primed bases of $V$ and $W$, respectively.

The most basic example of changing basis

Suppose $A$ is an $m \times n$ matrix. Then we have seen that the linear transformation $T(v) = Av$ has matrix $A$ with respect to the standard bases $e_1, \ldots, e_n$ and $e'_1, \ldots, e'_m$. The matrix of $T$ with respect to another pair of bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ is given by $C^{-1}AB$ where

$$B = \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} w_1 & \cdots & w_m \end{pmatrix}.$$

### SVD

Here is an algorithm that will always work for the SVD.

Suppose you are given an $m \times n$ matrix $A$.

1. Let $\lambda_1, \ldots, \lambda_i$ be the non-zero eigenvalues of $A^TA$ and $\lambda_{i+1}, \ldots, \lambda_n$ be the zero eigenvalues of $A^T A$. Choose corresponding ORTHONORMAL eigenvectors $v_1, \ldots, v_n$ for $A^T A$.
2. Let $\sigma_j = \sqrt{\lambda_j}$. Let $u_1, \ldots, u_i$ be given by $u_j = Av_j/\sigma_j$.
3. Let $u_{i+1}, \ldots, u_m$ be an ORTHONORMAL basis of $N(AA^T)$.
4. The SVD is

$$\begin{pmatrix} u_1 & \cdots & u_m \end{pmatrix} \Sigma \begin{pmatrix} v_1 & \cdots & v_n \end{pmatrix}^T$$

where $\Sigma$ is an $m \times n$ matrix with $(j, j)$-entry given by $\sigma_j$ and all other entries 0.

If $m < n$ it is a little quicker to do the following.

1. Let $\lambda_1, \ldots, \lambda_i$ be the non-zero eigenvalues of $AA^T$ and $\lambda_{i+1}, \ldots, \lambda_m$ be the zero eigenvalues of $AA^T$. Choose corresponding ORTHONORMAL eigenvectors $u_1, \ldots, u_m$ for $AA^T$.
2. Let $\sigma_j = \sqrt{\lambda_j}$. Let $v_1, \ldots, v_i$ be given by $v_j = A^Tu_j/\sigma_j$.
3. Let $v_{i+1}, \ldots, v_n$ be an ORTHONORMAL basis of $N(A^TA)$. 

2
Recitation 10 questions

Question 1

Let \( A = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 1 & 5 \end{pmatrix} \).

(a) Compute \( A(1,1,1)^T, A(1,1,0)^T, A(1,-1,0)^T \). Write each as a linear combination of the vectors \((1,1)^T\) and \((0,1)^T\).

(b) Find the matrix of the linear transformation \( T(x) = Ax \) with respect to the bases \((1,1,1)^T, (1,1,0)^T, (1,-1,0)^T\) and \((1,1)^T, (0,1)^T\) by using the definition of the matrix of a linear transformation.

(c) Find the matrix of the linear transformation \( T(x) = Ax \) with respect to the bases \((1,1,1)^T, (1,1,0)^T, (1,-1,0)^T\) and \((1,1)^T, (0,1)^T\) by using basis change matrices.

Question 2

Suppose \( A \) is a \( 3 \times 2 \) matrix with the property that
\[
A(4,5)^T = (1,2,3)^T \quad \text{and} \quad A(3,4)^T = (3,2,1)^T.
\]

(a) What is the matrix of \( T(x) = Ax \) with respect to the bases
\[
(4,5)^T, (3,4)^T \quad \text{and} \quad (1,2,3)^T, (3,2,1)^T, (0,1,0)^T.
\]

(b) Use the basis change formula to write the matrix you just calculated as \( C^{-1}AB \).

(c) What is \( A \)?

Question 3

Let \( V \) be the set of cubic polynomials
\[
V = \{ f(x) = ax^3 + bx^2 + cx + d : a, b, c, d \in \mathbb{R} \}.
\]

(a) Recall why \( V \) is a vector space; what happens to the coefficients under addition and scalar multiplication?

(b) Define \( T_1(f(x)) = f'(x) \). Is this a linear transformation \( V \longrightarrow V \)? Why?

(c) Define \( T_2(f(x)) = f(x+1) \). Is this a linear transformation \( V \longrightarrow V \)? Why?

(d) Define \( T_3(f(x)) = x^3 f(1/x) \). Is this a linear transformation \( V \longrightarrow V \)? Why?

(e) Recall that 1, \( x, x^2, x^3 \) is a basis for \( V \). Does this make sense to you?

(f) What is \( T_1(1), T_1(x), T_1(x^2), T_1(x^3) \)? What is the matrix of \( T_1 \) with respect to the basis \( 1, x, x^2, x^3 \) (using this as a basis for the domain and codomain)?

(g) What is the matrix of \( T_2 \) with respect to the basis \( 1, x, x^2, x^3 \)?

(h) What is the matrix of \( T_3 \) with respect to the basis \( 1, x, x^2, x^3 \)?

(i) What is the matrix of \( T_1 \) with respect to the basis \( 1, x-1, (x-1)(x-2), x(x^2 - \frac{9}{2}x + 6) \)?
Question 4

Let \( A = \begin{pmatrix} 1 & 2 \\ 3 & 6 \end{pmatrix} \).

1. Find the eigenvalues \( \lambda_1 \neq 0, \lambda_2 \) and unit eigenvectors \( v_1, v_2 \) of \( A^T A \).

2. Let \( \sigma_1 = \sqrt{\lambda_1}, u_1 = Av_1/\sigma_1 \). Verify that \( u_1 \) is a unit eigenvector for \( AA^T \) with eigenvalue \( \lambda_1 \).

3. Extend \( u_1 \) to an orthonormal basis \( u_1, u_2 \).

4. Check that

\[
A = (u_1|u_2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & 0 \end{pmatrix} (v_1|v_2)^T.
\]