Your PRINTED name is: ____________________

Please circle your recitation:

Grading

1

2

3

4

Total:

Each problem is 25 points, and each of its five parts (a)–(e) is 5 points.

In all problems, write all details of your solutions. Just giving an answer is not enough to get a full credit. Explain how you obtained the answer.
Problem 1.  (a) Do Gram-Schmidt orthogonalization for the vectors \[ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}. \]

(Find an orthogonal basis. Normalization is not required.)

(b) Find the $A = QR$ decomposition for the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$.

(c) Find the projection of the vector $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ onto the line spanned by the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(d) Find the projection of the vector $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ onto the plane $x + y + z = 0$ in $\mathbb{R}^3$.

(e) Find the least squares solution $\hat{x}$ for the system $\begin{pmatrix} 1 & -1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 10 \\ 0 \end{pmatrix}$. 
Problem 2. Let \( A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix} \).

(a) Calculate the determinant \( \det(A) \).

(b) Explain why \( A \) is an invertible matrix. Find the entry (2, 3) of the inverse matrix \( A^{-1} \).

(c) Notice that all sums of entries in rows of \( A \) are the same. Explain why this implies that \( (1, 1, 1)^T \) is an eigenvector of \( A \). What is the corresponding eigenvalue \( \lambda_1 \)?

(d) Find two other eigenvalues \( \lambda_2 \) and \( \lambda_3 \) of \( A \).

(e) Find the projection matrix \( P \) for the projection onto the column space of \( A \).
Problem 3.

(a) Calculate the area of the triangle on the plane $\mathbb{R}^2$ with the vertices $(1, 0), (0, 1), (3, 3)$ using determinants.

(b) Find all values of $x$ for which the matrix $A = \begin{pmatrix} 1 & x \\ 1 & 1 \end{pmatrix}$ has an eigenvalue equal to 2.

(c) Diagonalize the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(d) Calculate the power $B^{2014}$ of the matrix $B = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$.

(e) Let $Q$ be any matrix which is symmetric and orthogonal. Find $Q^{2014}$. Explain your answer.
Problem 4. Consider the Markov matrix \( A = \begin{pmatrix} 0 & 1/3 & 1/3 & 0 \\ 1/2 & 0 & 1/3 & 1/2 \\ 1/2 & 1/3 & 0 & 1/2 \\ 0 & 1/3 & 1/3 & 0 \end{pmatrix} \).

(a) Three of the eigenvalues of \( A \) are 1, 0, \(-1/3\). Find the fourth eigenvalue of \( A \).

(b) Find the determinant \( \det(A) \).

(c) Find the eigenvector of the transposed matrix \( A^T \) with the eigenvalue \( \lambda_1 = 1 \).

(d) Find the eigenvector of the matrix \( A \) with the eigenvalue \( \lambda_1 = 1 \). (Hint: Notice that nonzero entries in each column of \( A \) are the same.)

(e) Find the limit of \( A^k (1 \ 0 \ 0 \ 0)^T \) as \( k \to +\infty \).
If needed, you can use this extra sheet for your calculations.
If needed, you can use this extra sheet for your calculations.