18.06 Problem Set 10

due Monday, December 01, 2014, before 4:00 pm (sharp deadline) in Room E17-131

The due date is extended till Monday because of Thanksgiving vacation.

Problem 1. Let $A$ be a matrix with SVD $A = U\Sigma V^T$.
(a) Find an SVD of $A^T$.
(b) Find an SVD of $A^{-1}$ if $A$ is invertible.

Problem 2. Find an SVD for each of the following matrices:
(a) \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \),  
(b) \( \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \),  
(c) \( \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \).

Problem 3. A matrix can have several different SVDs. Let $A$ be an $n \times n$ matrix with $n$ different singular values $\sigma_1 > \sigma_2 > \cdots > \sigma_n$ (all $\sigma_i$’s are distinct and nonzero). Explain why all SVDs $A = U\Sigma V^T$ are obtained from each other by multiplying some columns of $U$ by $-1$ and simultaneously multiplying the same columns of $V$ by $-1$.

Problem 4. Find the set of all real numbers $u$ such that the matrix
\[
\begin{pmatrix}
1 & u & 0 \\
u & 1 & u \\
0 & u & 1
\end{pmatrix}
\]
is positive-definite.

Problem 5. If two symmetric matrices $A$ and $B$ are similar, then show that there exists an orthogonal matrix $M$ such that $B = MAM^{-1}$. (Hint: Diagonalize $A$ and $B$ and compare.)

Problem 6. (a) If at least one of the two $n \times n$ matrices $A$ and $B$ is invertible, then show that $AB$ is similar to $BA$.
(b) Is it true that $AB$ is similar to $BA$ for any two $n \times n$ matrices? (Hint: Find two matrices such that $AB$ is the zero matrix, but $BA$ is not zero.)
Problem 7. Consider the following operations on the (infinite dimensional) space of polynomials \( f(x) \). Which of them are linear transformations?

(a) \( T_1(f) = f(x) + 2 \).
(b) \( T_2(f) = 2f(x) \).
(c) \( T_3(f) = f(x+2) \).
(d) \( T_4(f) = f(2x) \).
(e) \( T_5(f) = f(x^2) \).
(f) \( T_6(f) = (f(x))^2 \).
(g) \( T_7(f) = f''(x) \).
(h) \( T_8(f) = x^2f(x) \).

Problem 8. Let \( V \) be the vector space of quadratic polynomials \( V = \{ ax^2 + bx + c \} \), and \( W \) be the vector space of cubic polynomials \( W = \{ \alpha x^3 + \beta x^2 + \gamma x + \delta \} \).

Let \( T : V \to W \) be the linear transformation given by

\[
T : f(x) \to (2 + x)f(x) + \int_0^x f(t) \, dt.
\]

Find the matrix of \( T \) with respect to the basis \( 1, x, x^2 \) of \( V \) and the basis \( 1, x, x^2, x^3 \) of \( W \).

Problem 9. Let \( T : V \to V \) be the linear transformation, whose input and output space \( V \) is the space of \( 2 \times 2 \) matrices, given by \( T(A) = A^T \). Find the matrix of the linear transformation \( T \) in the basis:

\[
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

Problem 10. Consider the subspace of vectors \((x, y, z)\) in \( \mathbb{R}^3 \) such that \( x+y+z = 0 \).
Find a basis for this subspace, and describe the linear transformation

\( T : (x, y, z) \mapsto (y, z, x) \)

on this subspace by a matrix with respect to this basis.