Problem set 6 solutions

Question 1
To find one vector in the plane, we set $x = 1$ and $z = 0$. This forces $y = -1$ and we get $(1, -1, 0)$. Any vector orthogonal to this vector has $x = y$ so set $x = y = 1$. The plane equation forces $z = -1$ and we get $(1, 1, -1)$. $(1, -1, 0)$ and $(1, 1, -1)$ are orthogonal and lie in the plane. $(1/\sqrt{2}, -1/\sqrt{2}, 0)$ and $(1/\sqrt{3}, 1/\sqrt{3}, -1/\sqrt{3})$ are orthonormal and lie in the plane.

Question 2
(a) Suppose $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are orthonormal and that $c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3 = \mathbf{0}$. Then 
$$c_1 = (c_1\mathbf{q}_1 + c_2\mathbf{q}_2 + c_3\mathbf{q}_3) \cdot \mathbf{q}_1 = 0.$$ 
Similarly, $c_2 = c_3 = 0$, so $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_3$ are linearly independent.

(b) Suppose $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are orthonormal.
Let $Q$ be the matrix with $i^{th}$ column given by $\mathbf{q}_i$. Then $Q^TQ = I_n$. A dependency relation
$$x_1\mathbf{q}_1 + \ldots + x_n\mathbf{q}_n = \mathbf{0}$$
can be written in matrix form as $Qx = \mathbf{0}$. But then $x = I_n x = (Q^TQ)x = Q^T(Qx) = \mathbf{0}$. The coefficients in the dependancy relation have to be zero, which means that $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are linearly independent.

Question 3
$p = \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}\mathbf{a}^T\mathbf{b} =$
\[
\frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.
\]
$e = \mathbf{b} - p = (-2, 0, 2)^T$.
$q_1 = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$, $q_2 = (-1/\sqrt{2}, 0, 1/\sqrt{2})$.

Question 4
$q_1 = \mathbf{a} = (1, 0, 0)^T$.
$q_2 = (\mathbf{b} - 2\mathbf{a})/3 = (0, 0, 1)^T$.
$q_3 = (\mathbf{c} - 2\mathbf{b})/5 = (0, 1, 0)^T$.

$Q = A \begin{pmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & -2/5 \\ 0 & 0 & 1/5 \end{pmatrix}$ and $A = Q \begin{pmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.
Question 5
(a)
It’s true for \((1 \times 1)\)-matrices and false for \((n \times n)\) matrices with \(n > 1\). Here’s a counterexample. Take \(A\) to be the matrix with \(-1\) in the \((1,1)\) entry and zeroes everywhere else. Then \(\det(I + A) = 0\). However, \(1 + \det(A) = 1\) when \(n > 1\).

(b)
True. By the product rule we have \(|ABC| = |A(BC)| = |A||BC| = |A||B||C|\).

(c)
True for \((1 \times 1)\) matrices or when \(A\) has determinant 0. If \(A\) is \((n \times n)\), where \(n > 1\), with nonzero determinant then \(|4A| = 4^n|A| \neq 4|A|\). For a really concrete counterexample to the truth of the general statement, note that when \(n > 1\) we have
\[|4I_n| = 4^n \neq 4 = 4|I_n|.

(d)
True for \((1 \times 1)\) matrices and false for \((n \times n)\) matrices with \(n > 1\).
\[
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
-\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]
has determinant 1. Let
\[A = \begin{pmatrix}
0 & 0 \\
0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]
For \((2n \times 2n)\)-matrices we can use the block matrices consisting of \(n\) copies of \(A\) along the diagonal, and \(n\)-copies of \(B\) along the diagonal, respectively, to find a counterexample. \(3 \times 3\) counterexample?

Question 6
If \(a = b\), \(b = c\), or \(c = a\), then
\[
\det \begin{pmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2
\end{pmatrix} = 0
\]
since two of the rows are the same, and thus the matrix is singular. So the formula \((b-a)(c-a)(c-b)\) is true in each of these cases. We now assume that \(a, b\) and \(c\) are distinct. Then
\[
\det \begin{pmatrix}
1 & a & a^2 \\
1 & b & b^2 \\
1 & c & c^2
\end{pmatrix} = \det \begin{pmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & c-a & c^2-a^2
\end{pmatrix} = \det \begin{pmatrix}
1 & a & a^2 \\
0 & b-a & b^2-a^2 \\
0 & 0 & d
\end{pmatrix} = (b-a)d,
\]
where \(d = (c^2-a^2) - \frac{c-a}{b-a}(b^2-a^2) = (c-a)((c+a)-(b+a)) = (c-a)(c-b)\), completing the proof.
Question 7

(a)
We take $A = 1$ and $B = x - \frac{(1,x)}{11}$. Then $\det(x, x) = \int_0^{2\pi} x \, dx = 2\pi^2$ so that $B = x - \pi$.

We take $C = x^2 - \frac{(1,x^2)}{11} x - \frac{(x-\pi,x^2)}{(x-\pi,x-\pi)}(x-\pi)$.

$$(1, x^2) = (x, x) = \int_0^{2\pi} x^2 \, dx = 8\pi^3/3$$ and $$x^2 = \int_0^{2\pi} x^3 \, dx = 4\pi^4.$$ 

So $(x-\pi,x-\pi) = (x, x) - 2\pi(1, x) + \pi^2(1, 1) = 8\pi^3/3 - 4\pi^3 + 2\pi^3 = 2\pi^3/3$ and $(x, x^2) = (x, x^2) - \pi(1, x^2) = 4\pi^4 - 8\pi^4/3 = 4\pi^4/3$.

Thus $C = x^2 - \frac{4\pi^2}{3}1 - 2\pi(x - \pi) = x^2 - 2\pi x + 2\pi^2/3$.

$$(x^2, x^2) = \int_0^{2\pi} x^4 \, dx = 52\pi^5/5.$$ 

So $(x^2 - 2\pi x + 2\pi^3/3, x^2 - 2\pi x + 2\pi^2/3) = (x^2, x^2 - 4\pi(x, x^2) + (4\pi^2/3)(1, x^2) + 4\pi^2(x, x) - (8\pi^3/3)(1, x) + (4\pi^4/9)(1, 1) = (32/5 - 16 + 32/9 + 32/3 - 16/3 + 8/9)\pi^5 = 8\pi^5/45.$

(b)
$q_1 = A/\sqrt{(A, A)} = 1/\sqrt{(1, 1)} = 1/\sqrt{2\pi}.
q_2 = B/\sqrt{(B, B)} = (x - \pi)/\sqrt{(x - \pi, x - \pi)} = (x - \pi)/\sqrt{2\pi^3/3}.
q_3 = C/\sqrt{(C, C)} = (x^2 - 2\pi x + 2\pi^2/3)/\sqrt{8\pi^5/45}.$

Question 8

(a)
$A_1$ is already reduced. $A_2$ reduces to $$\begin{pmatrix} 1 & 1 \\ 0 & 1 - x_1 \end{pmatrix}$$. $A_3$ reduces to $$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 - x_1 & 1 - x_1 \\ 0 & 0 & 1 - x_2 \end{pmatrix}$$.

So det $A_1 = 1$, det $A_2 = 1 - x_1$, det $A_3 = (1 - x_1)(1 - x_2)$ and the pivots are on the diagonal.

(b)
det $A_n = (1 - x_1)(1 - x_2) \cdots (1 - x_{n-1})$.

(c)
Let $e_i$ be the $1 \times n$ row vector consisting of $i$ zeroes, followed by $n - i$ ones.

For instance, $e_0 = (1, 1, \ldots, 1)$ and $e_1 = (0, 1, 1, \ldots, 1)$ and $e_{n-1} = (0, \ldots, 0, 0, 1)$.

Then $A_n$ reduces to a matrix with first row given by $e_0$, and where, for $i > 1$, the $i^{th}$ row is given by $(1 - x_{i-1})e_{i-1}$. The pivots are on the diagonal. We can calculate the determinant by multiplying the diagonal entries and this gives the formula we conjectured in part b).
Question 9

(a)
We already did this in question 5 part a). Work with \((n \times n)\)-matrices where \(n > 1\). Take \(A\) to be the matrix with \(-1\) in the \((1,1)\) entry and zeroes everywhere else and \(B\) to be the identity matrix. Then \(\det(A + B) = 0\). However, \(\det(A) + \det(B) = 1\).

(b)
We can construct a whole family of such matrices. Let \(a, b \in \mathbb{R}\), then
\[
\det \left( a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right) = a^2 + b^2 = \det \left( a \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) + \det \left( b \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right).
\]
For the purpose of answering the question, one may take \(a = b = 1\).

Question 10

See other document.