Your PRINTED name is: SOLUTIONS

Please circle your recitation:

(1) M 2 2-131 A. Osorno  
    Grading  
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    1  

(2) M 3 2-131 A. Osorno  
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(3) M 3 2-132 A. Pissarra Pires  
    ________  
    2  

(4) T 11 2-132 K. Meszaros  
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(5) T 12 2-132 K. Meszaros  
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(6) T 1 2-132 Jerin Gu  
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(7) T 2 2-132 Jerin Gu  
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Total:
Problem 1 (10 points)

Let  
\[
A = \begin{pmatrix}
3 & 2 & 1 & 1 \\
6 & 6 & 3 & 3 \\
3 & 4 & 2 & 2
\end{pmatrix}.
\]

(a) Calculate the dimensions of the 4 fundamental subspaces associated with \( A \).

(b) Give a basis for each of the 4 fundamental subspaces.

(c) Find the complete solution of the system  
\[
A x = \begin{pmatrix}
1 \\
3 \\
2
\end{pmatrix}.
\]

Solution 1

(a) The rank is 2, so the dimensions are:

\[ C(A) \sim r = 2 \]
\[ C(A^T) \sim r = 2 \]
\[ N(A) \sim n - r = 4 - 2 = 2 \]
\[ N(A^T) \sim m - r = 3 - 2 = 1. \]

(b) We can get these by elimination or by inspection:

\[ C(A) \sim \begin{pmatrix} 3 & 6 & 3 \end{pmatrix}^T, \begin{pmatrix} 2 & 6 & 4 \end{pmatrix}^T \]
\[ C(A^T) \sim \begin{pmatrix} 3 & 2 & 1 \end{pmatrix}^T, \begin{pmatrix} 6 & 3 & 3 \end{pmatrix}^T \]
\[ N(A) \sim \begin{pmatrix} 0 & -1/2 & 1 & 0 \end{pmatrix}^T, \begin{pmatrix} 0 & -1/2 & 0 & 1 \end{pmatrix}^T \]
\[ N(A^T) \sim \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}^T \]

(c) \( x = x_{\text{particular}} + x_{\text{nullspace}} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix} + c_1 \begin{pmatrix} 0 \\ -1/2 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ -1/2 \\ 0 \end{pmatrix} \).
Problem 2 (10 points)

Consider the system of linear equations:

\[
\begin{align*}
  x + y + z &= 1 \\
  2x + z &= 2 \\
  -x + y + az &= b
\end{align*}
\]

In parts (a)–(c) below circle correct answers. Explain your answers.

(a) For \( a = 1, b = -1 \), the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(b) For \( a = 0, b = 1 \), the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(c) For \( a = 0, b = -1 \), the system has:

(1) exactly one solution

(2) infinitely many solutions

(3) no solutions

(d) Solve the system for \( a = b = 1 \).

Solution 2

If we eliminate the augmented matrix we get

\[
\begin{pmatrix}
  1 & 1 & 1 & 1 \\
  0 & -2 & -1 & 0 \\
  0 & 0 & a & b + 1
\end{pmatrix}.
\]

(a) **Exactly one solution:** The matrix is invertible.

(b) **No solutions:** Get a row of zeroes in the matrix with no zero in the augmented column.

(c) **Infinitely many solutions:** Get a row of zeroes with a zero in the augmented column.

(d) Using back substitution we get \( x = \begin{pmatrix} 0 & -1 & 2 \end{pmatrix}^T \).
Problem 3 (10 points)

Let $L$ be the line in $\mathbb{R}^3$ spanned by the vector $(1, 1, 1)^T$. Let $P$ be the projection matrix for the projection onto the line $L$.

(a) What are the eigenvalues of the matrix $P$? (Indicate their multiplicities.)

(b) Find an orthonormal basis of the orthogonal complement $L^\perp$ to the line $L$.

(c) Calculate the projection of the vector $(1, 2, 3)^T$ onto the line $L$.

(d) Calculate the projection of the vector $(1, 2, 3)^T$ onto the orthogonal complement $L^\perp$.

Solution 3

(a) $P$ is a projection matrix onto a subspace of dimension 1, so the eigenvalues are 1, 0, 0.

(b) \[
\begin{pmatrix}
\frac{1}{\sqrt{2}} \\
0 \\
-\frac{1}{\sqrt{2}}
\end{pmatrix},
\begin{pmatrix}
\frac{1}{\sqrt{6}} \\
-\frac{2}{\sqrt{6}} \\
\frac{1}{\sqrt{6}}
\end{pmatrix}.
\]

(c) $p = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

(d) The projection onto $L^\perp$ is $b - p = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. 
Problem 4 (10 points)

Let \( A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 2 & 1 \end{pmatrix} \).

In parts (a)-(c) below circle correct answers. Explain your answers.

(a) The matrix \( A \) is singular: True   False

(b) The matrix \( A + 2I \) is singular: True   False

(c) The matrix \( A \) is positive definite: True   False

(d) Find all eigenvalues of \( A \) and the corresponding eigenvectors.

(e) Find an orthogonal matrix \( Q \) and a diagonal matrix \( \Lambda \) such that \( A = Q\Lambda Q^T \).

(f) Solve the system of differential equations \( \frac{d\mathbf{u}(t)}{dt} = A\mathbf{u}(t), \mathbf{u}(0) = (1, 0, 0)^T \).

Solution 4

(a) True \( \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T \) is in the nullspace.

(b) True \( \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T \) is in the nullspace.

(c) False The matrix is singular so has 0 as eigenvalue.

(d) \( A \) is singular, so 0 is an eigenvalue with eigenvector \( \begin{pmatrix} 1 & -2 & 1 \end{pmatrix}^T \).

\( A + 2I \) is singular, so -2 is an eigenvalue with eigenvector \( \begin{pmatrix} 1 & 0 & -1 \end{pmatrix}^T \).

We can get the last eigenvalue by looking a the trace: 6. The eigenvector is \( \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T \).

\( A = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} 1/\sqrt{6} & -2/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{pmatrix} \).

\( \mathbf{u}(t) = e^{At}\mathbf{u}(0) = Qe^{\Lambda t}Q^T\mathbf{u}(0) = \frac{1}{6} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \frac{1}{2} e^{-2t} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \frac{1}{3} e^{6t} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} . \)
Problem 5 (10 points)

Let \( A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 \end{pmatrix} \).

(a) What is the rank of \( A \)?

(b) Calculate the matrix \( A^T A \). Find all its eigenvalues (with multiplicities).

(c) Calculate the matrix \( A A^T \). Find all its eigenvalues (with multiplicities).

(d) Find the matrix \( \Sigma \) in the singular value decomposition \( A = U \Sigma V^T \).

Solution 5

(a) \( 1 \)

(b) \( A^T A = \begin{pmatrix} 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \\ 14 & 14 & 14 & 14 \end{pmatrix} \). It was rank 1 so the eigenvalues are 56, 0, 0, 0.

(c) \( A A^T = \begin{pmatrix} 4 & 8 & 12 \\ 8 & 16 & 24 \\ 12 & 24 & 36 \end{pmatrix} \). It was rank 1 so the eigenvalues are 56, 0, 0.

(d) \( \Sigma = \begin{pmatrix} \sqrt{56} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \).
Problem 6 (10 points)

Let $A_n$ be the tridiagonal $n \times n$-matrix with 2's on the main diagonal, 1's immediately above the main diagonal, 3's immediately below the main diagonal, and 0's everywhere else:

$$A_n = \begin{pmatrix}
2 & 1 & 0 & 0 & \cdots & 0 \\
3 & 2 & 1 & 0 & \ddots & 0 \\
0 & 3 & 2 & 1 & \ddots & 0 \\
0 & 0 & 3 & 2 & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 2
\end{pmatrix},$$

(a) Express the determinant $\det(A_n)$ in terms of $\det(A_{n-1})$ and $\det(A_{n-2})$.

(b) Explicitly calculate $\det(A_n)$, for $n = 1, \ldots, 6$.

Solution 6

(a) Using cofactors twice we get $\det(A_n) = 2 \det(A_{n-1}) - 3 \det(A_{n-2})$.

(b) $\det(A_1) = \det[2] = 2$.

$\det(A_2) = \det \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} = 1.$

$\det(A_3) = 2 \cdot 1 - 3 \cdot 2 = -4.$

$\det(A_4) = 2 \cdot (-4) - 3 \cdot 1 = -11.$

$\det(A_5) = 2 \cdot (-11) - 3 \cdot (-4) = -10.$

$\det(A_6) = 2 \cdot (-10) - 3 \cdot (-11) = 13.$
Problem 7 (10 points)

Calculate the determinant of the following $6 \times 6$-matrix:

$$A = \begin{pmatrix}
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1
\end{pmatrix}$$

Solution 7

This determinant could be computed using cofactors or doing row operations to simplify and then cofactors. It could also be computed as follows.

$$A = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0
\end{pmatrix}.$$

$P$ is a permutation matrix with determinant $-1$.

$$B = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{pmatrix} - I.$$

The eigenvalues of the matrix with all 1’s are 6, 0, 0, 0, 0, 0 so the eigenvalues of $B$ are 5, -1, -1, -1, -1, -1, so the determinant of $B$ is $-5$.

Thus the determinant of $A$ is 5.
**Problem 8 (10 points)**

(a) Calculate $A^{100}$ for $A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

(b) Calculate $B^{100}$ for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

(c) What will happen with the house (shown below) when we apply the linear transformation $T(v) = Bv$ for $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Will it be dilated? Draw the picture of the transformed house.

**Solution 8**

(a) $A^2 = \begin{pmatrix} 25 & 0 \\ 0 & 25 \end{pmatrix}$ so $A^{100} = \begin{pmatrix} 5^{100} & 0 \\ 0 & 5^{100} \end{pmatrix}$.

(b) $B^4 = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$ so $B^{100} = \begin{pmatrix} (-4)^{25} & 0 \\ 0 & (-4)^{25} \end{pmatrix} = \begin{pmatrix} -4^{25} & 0 \\ 0 & -4^{25} \end{pmatrix}$.

(c) From part (b) we see that $B^4$ is stretching by 4 and rotating by $\pi$. Thus $B$ is rotating by $\pi/4$ and stretching by $\sqrt{2}$. 