18.06 Problem Set 3

Due Wednesday, 25 February 2008 at 4pm in 2-106.

1. Consider the matrix
   \[ A = \begin{pmatrix} 1 & 2 & 1 & 4 & 1 \\ 2 & 6 & 3 & 11 & 1 \\ 1 & 4 & 2 & 7 & 0 \end{pmatrix} \]

   (a) Reduce \( A \) to echelon form \( U \), find a special solution for each free variable, and hence describe all solutions to \( Ax = 0 \).
   
   (b) By further row operations on \( U \), find the reduced echelon form \( R \).
   
   (c) True or false: \( N(R) = N(U) \)?
   
   (d) True or false: \( C(A) = C(U) \)?

2. If you do column elimination steps (instead of row eliminations) on a matrix \( A \) to get some other matrix \( U \) (like in problem 6 of pset 1), does \( N(A) = N(U) \)? Come up with a counter-example if false, or give an explanation why this should always hold if true.

3. Suppose that column 3 of a \( 4 \times 6 \) matrix is all zero. Then \( x_3 \) must be a ___________ variable. Give one special solution for this matrix.

4. Fill in the missing numbers to make the matrix \( A \) rank 1, rank 2, and rank 3. (i.e. your solution should be three matrices).
   \[ A = \begin{pmatrix} \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ \\ \_ & \_ & \_ & \_ & \_ & \_ \end{pmatrix} \]

5. Suppose \( A \) and \( B \) have the same reduced echelon form \( R \). Therefore \( A \) equals a/an ___________ matrix multiplying \( B \) on the ________ (left or right).

6. Write the complete solution (i.e. a particular solution plus all nullspace vectors) to the system:
   \[ \begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} . \]

7. Explain why these statements are all false by giving a counter-example for each:
   
   (a) A system \( Ax = b \) has at most one particular solution.
   
   (b) A system \( Ax = b \) has at least one particular solution.
   
   (c) If there is only one special solution \( x_n \) in the nullspace and there exists some particular solution \( x_p \), then the complete solution to \( Ax = b \) is any linear combination of \( x_p \) and \( x_n \).
   
   (d) If \( A \) is invertible then there is no solution \( x_n \) the nullspace.
   
   (e) The solution \( x_p \) with all free variables set to zero is the “shortest” solution (minimizing \( ||x|| \)).

8. If \( A \) is a \( 3 \times 7 \) matrix, its largest possible rank is ________. In this case, there is a pivot in every _________ of \( U \) and \( R \), the solution to \( Ax = b \) ________ (always exists or is unique), and the column space of \( A \) is _________. Construct an example of such a matrix \( A \).

9. If \( A \) is a \( 6 \times 3 \) matrix, its largest possible rank is ________. In this case, there is a pivot in every _________ of \( U \) and \( R \), the solution to \( Ax = b \) ________ (always exists or is unique), and the nullspace of \( A \) is _________. Construct an example of such a matrix \( A \).
10. Find the rank of $A$, $A^T A$, and $AA^T$, for $A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 2 \end{pmatrix}$.

11. Choose three independent columns of $A = \begin{pmatrix} 2 & 3 & 4 & 1 \\ 4 & 12 & 15 & 2 \\ 0 & 0 & 0 & 9 \\ 0 & 6 & 7 & 0 \end{pmatrix}$. Then choose a different three independent columns. Explain whether either of these choices forms a basis for $C(A)$.

12. Find a basis for the space of $2 \times 3$ matrices whose nullspace contains $(1, 2, 0)$.

13. Make the matrix $A = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}$ in Matlab by the command:

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>> A = [2 1; 6 3]
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Then compute $b = Ax$ for 100 random $x$ vectors by the command:

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>> br = A * rand(2, 100);
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Plot these $b$ vectors as black dots by the commands:

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>> plot(br(1,:), br(2,:), 'k.')
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What is the pattern, and why?