1. (12 points) This question is about the matrix

\[ A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 2 & 4 & 1 & 4 \\ 3 & 6 & 3 & 9 \end{bmatrix}. \]

(a) Find a lower triangular \( L \) and an upper triangular \( U \) so that \( A = LU \).

Answer:

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

(b) Find the reduced row echelon form \( R = rref(A) \). How many independent columns in \( A \)?

Answer: 2

\[ R = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U \text{ in this example.} \]

(c) Find a basis for the nullspace of \( A \).

Answer:

\[ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix} \]
(d) If the vector $b$ is the sum of the four columns of $A$, write down the complete solution to $Ax = b$.

**Answer:**

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$
2. (11 points) This problem finds the curve \( y = C + D 2^t \) which gives the best least squares fit to the points \((t, y) = (0, 6), (1, 4), (2, 0)\).

(a) Write down the 3 equations that would be satisfied if the curve went through all 3 points.

Answer:

\[
\begin{align*}
C + 1D &= 6 \\
C + 2D &= 4 \\
C + 4D &= 0
\end{align*}
\]

(b) Find the coefficients \( C \) and \( D \) of the best curve \( y = C + D 2^t \).

Answer:

\[
\begin{align*}
A^T A &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \\
A^T b &= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}
\end{align*}
\]

Solve \( A^T A \hat{x} = A^T b \) :

\[
\begin{bmatrix} 3 & 7 \\ 7 & 21 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 10 \\ 14 \end{bmatrix}
\]

\[\text{gives } \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 21 & -7 \\ -7 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 14 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix}.\]

(c) What values should \( y \) have at times \( t = 0, 1, 2 \) so that the best curve is \( y = 0 \)?

Answer:

The projection is \( p = (0, 0, 0) \) if \( A^T b = 0 \). In this case, \( b \) = values of \( y = c(2, -3, 1) \).
3. **(11 points)** Suppose $Av_i = b_i$ for the vectors $v_1, \ldots, v_n$ and $b_1, \ldots, b_n$ in $\mathbb{R}^n$. Put the $v$'s into the columns of $V$ and put the $b$'s into the columns of $B$.

(a) Write those equations $Av_i = b_i$ in matrix form. What condition on which vectors allows $A$ to be determined uniquely? Assuming this condition, find $A$ from $V$ and $B$.

**Answer:**

$A [v_1 \cdots v_n] = [b_1 \cdots b_n]$ or $AV = B$. Then $A = BV^{-1}$ if the $v$'s are independent.

(b) Describe the column space of that matrix $A$ in terms of the given vectors.

**Answer:**

The column space of $A$ consists of all linear combinations of $b_1, \ldots, b_n$.

(c) What additional condition on which vectors makes $A$ an invertible matrix? Assuming this, find $A^{-1}$ from $V$ and $B$.

**Answer:**

If the $b$'s are independent, then $B$ is invertible and $A^{-1} = VB^{-1}$. 
4. (11 points)

(a) Suppose $x_k$ is the fraction of MIT students who prefer calculus to linear algebra at year $k$.

The remaining fraction $y_k = 1 - x_k$ prefers linear algebra.

At year $k + 1$, $1/5$ of those who prefer calculus change their mind (possibly after taking 18.03). Also at year $k + 1$, $1/10$ of those who prefer linear algebra change their mind (possibly because of this exam).

Create the matrix $A$ to give 
\[
\begin{bmatrix}
  x_{k+1} \\
  y_{k+1}
\end{bmatrix} = A
\begin{bmatrix}
  x_k \\
  y_k
\end{bmatrix}
\]

and find the limit of $A^k$ as $k \to \infty$.

**Answer:**

\[
A = \begin{bmatrix}
  .8 & .1 \\
  .2 & .9
\end{bmatrix}.
\]

The eigenvector with $\lambda = 1$ is 
\[
\begin{bmatrix}
  1/3 \\
  2/3
\end{bmatrix}.
\]

This is the steady state starting from 
\[
\begin{bmatrix}
  1 \\
  0
\end{bmatrix}.
\]

\[
\frac{2}{3}
\]

of all students prefer linear algebra! I agree.

(b) Solve these differential equations, starting from $x(0) = 1, \ y(0) = 0$:

\[
\frac{dx}{dt} = 3x - 4y \quad \frac{dy}{dt} = 2x - 3y.
\]

**Answer:**

\[
A = \begin{bmatrix}
  3 & -4 \\
  2 & -3
\end{bmatrix}.
\]

has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with eigenvectors $x_1 = (2, 1)$ and $x_2 = (1, 1)$.

The initial vector $(x(0), y(0)) = (1, 0)$ is $x_1 - x_2$.

So the solution is $(x(t), y(t)) = e^t(2, 1) + e^{-t}(1, 1)$. 
(c) For what initial conditions \[ \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} \] does the solution \[ \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \] to this differential equation lie on a single straight line in \( \mathbb{R}^2 \) for all \( t \)?

**Answer:**

If the initial conditions are a multiple of either eigenvector \((2,1)\) or \((1,1)\), the solution is at all times a multiple of that eigenvector.
5. (11 points)

(a) Consider a 120° rotation around the axis \(x = y = z\). Show that the vector \(i = (1, 0, 0)\) is rotated to the vector \(j = (0, 1, 0)\). (Similarly \(j\) is rotated to \(k = (0, 0, 1)\) and \(k\) is rotated to \(i\).) How is \(j - i\) related to the vector \((1, 1, 1)\) along the axis?

**Answer:**

\[
-j - i = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}
\]

is orthogonal to the axis vector \(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\).

So are \(k - j\) and \(i - k\). By symmetry the rotation takes \(i\) to \(j\), \(j\) to \(k\), \(k\) to \(i\).

(b) Find the matrix \(A\) that produces this rotation (so \(Av\) is the rotation of \(v\)). Explain why \(A^3 = I\). What are the eigenvalues of \(A\)?

**Answer:**

\(A^3 = I\) because this is three 120° rotations (so 360°). The eigenvalues satisfy \(\lambda^3 = 1\) so \(\lambda = 1, e^{2\pi i/3}, e^{-2\pi i/3} = e^{4\pi i/3}\).

(c) If a 3 by 3 matrix \(P\) projects every vector onto the plane \(x + 2y + z = 0\), find three eigenvalues and three independent eigenvectors of \(P\). No need to compute \(P\).

**Answer:** The plane is perpendicular to the vector \((1, 2, 1)\). This is an eigenvector of \(P\) with \(\lambda = 0\). The vectors \((-2, 1, 0)\) and \((1, -1, 1)\) are eigenvectors with \(\lambda = 0\).
6. **(11 points)** This problem is about the matrix

\[
A = \begin{bmatrix}
1 & 2 \\
2 & 4 \\
3 & 6
\end{bmatrix}.
\]

(a) Find the eigenvalues of \(A^T A\) and also of \(AA^T\). For both matrices find a complete set of orthonormal eigenvectors.

**Answer:**

\[
A^T A = \begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 2 \\
2 & 4 \\
3 & 6
\end{bmatrix} = \begin{bmatrix}
14 & 28 \\
28 & 56
\end{bmatrix}
\]

has \(\lambda_1 = 70\) and \(\lambda_2 = 0\) with eigenvectors \(x_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}\) and \(x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}\).

\[
AA^T = \begin{bmatrix}
1 & 2 \\
2 & 4 \\
3 & 6
\end{bmatrix}
\begin{bmatrix}
1 & 2 & 3 \\
2 & 4 & 6
\end{bmatrix} = \begin{bmatrix}
5 & 10 & 15 \\
10 & 20 & 30 \\
15 & 30 & 45
\end{bmatrix}
\]

has \(\lambda_1 = 70\), \(\lambda_2 = 0\), \(\lambda_3 = 0\) with

\[
x_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\] and \(x_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}\) and \(x_3 = \frac{1}{\sqrt{70}} \begin{bmatrix} 3 \\ 6 \\ -5 \end{bmatrix}\).

(b) If you apply the Gram-Schmidt process (orthonormalization) to the columns of this matrix \(A\), what is the resulting output?

**Answer:**

Gram-Schmidt will find the unit vector

\[
q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
\]

But the construction of \(q_2\) fails because column 2 = 2 (column 1).
(c) If $A$ is any $m \times n$ matrix with $m > n$, tell me why $AA^T$ cannot be positive definite. Is $A^TA$ always positive definite? (If not, what is the test on $A$?)

**Answer**

$AA^T$ is $m \times m$ but its rank is not greater than $n$ (all columns of $AA^T$ are combinations of columns of $A$). Since $n < m$, $AA^T$ is singular.

$A^TA$ is positive definite if $A$ has full column rank $n$. (Not always true, $A$ can even be a zero matrix.)
7. **(11 points)** This problem is to find the determinants of

\[
A = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix} \quad
B = \begin{bmatrix}
0 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix} \quad
C = \begin{bmatrix}
x & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]

(a) Find \( \det A \) and give a reason.

**Answer:**
\( \det A = 0 \) because two rows are equal.

(b) Find the cofactor \( C_{11} \) and then find \( \det B \). This is the volume of what region in \( \mathbb{R}^4 \)?

**Answer:**
The cofactor \( C_{11} = -1 \). Then \( \det B = \det A - C_{11} = 1 \). This is the volume of a box in \( \mathbb{R}^4 \) with edges = rows of \( B \).

(c) Find \( \det C \) for any value of \( x \). You could use linearity in row 1.

**Answer:**
\( \det C = xC_{11} + \det B = -x + 1 \). Check this answer (zero), for \( x = 1 \) when \( C = A \).
8. (11 points)

(a) When $A$ is similar to $B = M^{-1} AM$, prove this statement:
If $A^k \to 0$ when $k \to \infty$, then also $B^k \to 0$.

**Answer:**
$A$ and $B$ have the same eigenvalues. If $A^k \to 0$ then all $|\lambda| < 1$. Therefore $B^k \to 0$.

(b) Suppose $S$ is a fixed invertible 3 by 3 matrix.
This question is about all the matrices $A$ that are diagonalized by $S$, so that $S^{-1} AS$ is diagonal. Show that these matrices $A$ form a subspace of 3 by 3 matrix space. (Test the requirements for a subspace.)

**Answer:**
If $A_1$ and $A_2$ are in the space, they are diagonalized by $S$. Then $S^{-1}(cA_1 + dA_2)S$ is diagonal + diagonal = diagonal.

(c) Give a basis for the space of 3 by 3 diagonal matrices. Find a basis for the space in part (b) — all the matrices $A$ that are diagonalized by $S$.

**Answer:**
A basis for the diagonal matrices is

\[
D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad D_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad D_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}
\]

Then $SD_1 S^{-1}, SD_2 S^{-1}, SD_3 S^{-1}$ are all diagonalized by $S$: a basis for the subspace.
9. (11 points) This square network has 4 nodes and 6 edges. On each edge, the direction of positive current $w_i > 0$ is from lower node number to higher node number. The voltages at the nodes are $(v_1, v_2, v_3, v_4)$.

Answer:

(a) Write down the incidence matrix $A$ for this network (so that $Av$ gives the 6 voltage differences like $v_2 - v_1$ across the 6 edges). What is the rank of $A$? What is the dimension of the nullspace of $A^T$?

Answer:

$$A = \begin{bmatrix}
-1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
-1 & 0 & 0 & 1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & -1 & 1 \\
\end{bmatrix}$$

has rank $r = 3$. The nullspace of $A^T$ has dimension $6 - 3 = 3$. 
(b) Compute the matrix $A^T A$. What is its rank? What is its nullspace?

**Answer:**

$$A^T A = \begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

has rank 3 like $A$. The nullspace is the line through $(1, 1, 1, 1)$.

(c) Suppose $v_1 = 1$ and $v_4 = 0$. If each edge contains a unit resistor, the currents $(w_1, w_2, w_3, w_4, w_5, w_6)$ on the 6 edges will be $w = -Av$ by Ohm’s Law. Then Kirchhoff’s Current Law (flow in = flow out at every node) gives $A^T w = 0$ which means $A^T A v = 0$. Solve $A^T A v = 0$ for the unknown voltages $v_2$ and $v_3$. Find all 6 currents $w_1$ to $w_6$. How much current enters node 4?

**Answer:**

*Note:* As stated there is no solution (my apologies!). All solutions to $A^T A v = 0$ are multiples of $(1, 1, 1, 1)$ which rules out $v_1 = 1$ and $v_4 = 0$.

*Intended problem:* I meant to solve the reduced equations using $KCL$ only at nodes 2 and 3. In fact symmetry gives $v_2 = v_3 = \frac{1}{2}$. Then the currents are $w_1 = w_2 = w_5 = w_6 = \frac{1}{2}$ around the sides and $w_3 = 1$ and $w_4 = 0$ (symmetry). So $w_3 + w_5 + w_6 = \frac{1}{2}$ is the total current into node 4.