1. (a) By elimination find the rank of $A$ and the pivot columns of $A$ (in its column space):

$$A = \begin{bmatrix} 1 & 2 & 1 & 4 \\ 3 & 6 & 3 & 9 \\ 2 & 4 & 2 & 9 \end{bmatrix}.$$ 

(b) Find the special solutions to $Ax = 0$ and then find all solutions to $Ax = 0$.

(c) For which number $b_3$ does $Ax = \begin{bmatrix} 3 \\ 9 \\ b_3 \end{bmatrix}$ have a solution?

Write the complete solution $x$ (the general solution) with that value of $b_3$.

(a) \[
\begin{bmatrix}
\frac{1}{3} & 2 & \frac{1}{4} \\
2 & 4 & 2 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 1 & b_3 - 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]

t = rank(A) = 2 , pivot columns are \[\begin{bmatrix} 1 \\ 2 \end{bmatrix}\] and \[\begin{bmatrix} 4 \\ 9 \end{bmatrix}\]

(b) Special solutions: \[\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\] \[N(A) = \begin{bmatrix} c_1 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \end{bmatrix}\]

All solutions: \[c_1, c_2 \in \mathbb{R}\]

(c) \[
\begin{bmatrix}
3 & 6 & 3 & 9 \\
2 & 4 & 2 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 0 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 1 & 4 \\
0 & 0 & 0 & 3
\end{bmatrix}
\]

Hence to have a solution we need \[b_3 - 6 = 0 \Rightarrow b_3 = 6\]

For this value of $b_3$, a particular solution is given by \[x_p = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}\]

complete solution: \[x_c = x_p + x_n = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}\]
2. Suppose $A$ is a 3 by 5 matrix and the equation $Ax = b$ has a solution for every $b$. What are (a)(b)(c)(d)? (If you don’t have enough information to answer, tell as much about the answer as you can.)

(a) Column space of $A$: Since $Ax = b \implies b = x_1 A_1 + \ldots + x_n A_n$
where $A_1, \ldots, A_n$ are the columns of $A$, every $b$ in $\mathbb{R}^3$ is in $C(A)$.
So $C(A) = \mathbb{R}^3$.

(b) Nullspace of $A$: Since $C(A) = \mathbb{R}^3$, we have $r = 3$ and therefore
$\# \text{ free variables} = \# \text{ special solutions} = n - r = 5 - 3 = 2$.
Hence $N(A)$ is a plane in $\mathbb{R}^5$.

(c) Rank of $A$
By (a) $C(A) = \mathbb{R}^3$ therefore $r = 3$.
We can also argue by saying that $r = m$ or we have

(d) Rank of the 6 by 5 matrix $B = \begin{bmatrix} A \\ A \end{bmatrix}$.
We can use elimination to obtain
\[
\begin{bmatrix} A \\ A \end{bmatrix} \rightarrow \begin{bmatrix} A \\ 0 \end{bmatrix}.
\]

But $\begin{bmatrix} A \\ 0 \end{bmatrix}$ has rank 3. Therefore $\text{rank}(B) = 3$. 
3. (a) When an odd permutation matrix $P_1$ multiplies an even permutation matrix $P_2$, the product $P_1P_2$ is odd (EXPLAIN WHY).

$P_1$ applies an odd number of row exchanges to $I$ and $P_2$ applies an even number. Hence $P_1P_2$ applies $\text{odd} + \text{even} = \text{odd}$ number of row exchanges.

(b) If the columns of $B$ are vectors in the nullspace of $A$, then $AB$ is 0 matrix (EXPLAIN WHY). Let $B = \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix}$, where $B_1, \ldots, B_k$ are the columns of $B$. Since each $B_i$ is in $N(A)$, we have $AB_i = 0$.

Then $AB = \begin{bmatrix} AB_1 & \cdots & AB_k \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \end{bmatrix} = 0$

(c) If $c = 0$, factor this matrix into $A = LU$ (lower triangular times upper triangular):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}$$

(d) That matrix $A$ is invertible unless $c = \frac{-21}{21}$.

(b) $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 4 & 9 \\ 1 & 8 & c \end{bmatrix}$

$-R_1+R_2 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 6 & c-3 \end{bmatrix}$

$-3R_2+R_3 \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & c-21 \end{bmatrix}$

Hence $A$ is invertible unless $c-21 = 0 \Rightarrow c = 21$

(a) When $c = 0$ we get $A = LU$,

where $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 6 \\ 0 & 0 & 0 \end{bmatrix}$