1. (a) Find two eigenvalues and eigenvectors of
\[ A = \begin{bmatrix} 2 & 3 \\ 0 & 5 \end{bmatrix} \]

(b) Express any vector \( u_0 = \begin{bmatrix} a \\ b \end{bmatrix} \) as a combination of the eigenvectors.

(c) What is the solution \( u(t) \) to \( \frac{du}{dt} = Au \) starting from \( u(0) = u_0 \)?

(d) Find a formula \( u_k = \ldots \) for the solution to \( u_{k+1} = Au_k \) which starts from that vector \( u_0 \). Set \( k = -1 \) to find \( A^{-1}u_0 \).
2. This problem is about the matrix

\[ A = \begin{bmatrix} \sqrt{2} & 1 \\ 0 & \sqrt{2} \end{bmatrix}. \]

(a) Find all eigenvectors of \( A \). Exactly why is it impossible to diagonalize \( A \) in the form \( A = SAS^{-1} \)?

(b) Find the matrices \( U, \Sigma, V^T \) in the Singular Value Decomposition \( A = U \Sigma V^T \).

Tell me two orthogonal vectors \( u_1, u_2 \) in the plane so that \( Au_1 \) and \( Au_2 \) are also orthogonal.

(c) Find a matrix \( B \) that is similar to \( A \) (but different from \( A \)).

Show that \( A \) and \( B \) meet the requirement to be similar (\textit{what is it?}).
3. Suppose $A$ is a real $m$ by $n$ matrix.

(a) Prove that the symmetric matrix $A^T A$ has the property $x^T (A^T A) x \geq 0$ for every vector $x$ in $\mathbb{R}^n$. Explain each step in your reason.

(b) According to part (a), the matrix $A^T A$ is positive semidefinite at least — and possibly positive definite. Under what condition on $A$ is $A^T A$ positive definite?

(c) If $m < n$ prove that $A^T A$ is not positive definite.