18.06 Solutions to PSet 8

6.4:

5: \( Q = \frac{1}{3} \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -1 \\ 2 & -2 & -1 \end{bmatrix} \). The columns of \( Q \) are unit eigenvectors of \( A \)
Each unit eigenvector could be multiplied by \(-1\)

7: (a) (a) \( \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) has \( \lambda = -1 \) and 3
(b) The pivots have the same signs as the \( \lambda \)'s
(c) trace = \( \lambda_1 + \lambda_2 = 2 \), so \( A \) can’t have two negative eigenvalues.
(\( \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \) has \( \lambda = -1 \)
and 3
(b) The pivots have the same signs as the \( \lambda \)'s
(c) trace = \( \lambda_1 + \lambda_2 = 2 \), so \( A \) can’t have two negative eigenvalues.

16: (a) If \( Ax = \lambda y \) and \( A^T y = \lambda z \) then \( B[y: -z] = [A(ax; A^T y)] = -\lambda[y: -z] \).
So \(-\lambda \) is also an eigenvalue of \( B \). (b) \( A^T A = A^T (\lambda y) = \lambda^2 z \).
(c) \( \lambda = -1, -1, 1, 1 \); \( x_1 = (1, 0, -1, 0), x_2 = (0, 1, 0, -1), x_3 = (1, 0, 1, 0), x_4 = (0, 1, 0, 1) \).

23: \( A \) is invertible, orthogonal, permutation, diagonalizable, Markov; \( B \) is projection, diagonalizable, Markov. \( A \) allows \( QR, SAS^{-1}, QAQ^T; B \) allows \( SAS^{-1} \) and \( QAQ^T \).

6.5:

8: \( A = \begin{bmatrix} 3 & 6 \\ 6 & 16 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \). Pivots 3, 4 outside squares, \( \ell_{ij} \) inside.

12: \( A \) is positive definite for \( c > 1 \); determinants \( c, c^2 - 1, \) and \((c - 1)^2(c + 2) > 0 \). \( B \) is never positive definite (determinants \( d - 4 \) and \(-4d + 12 \) are never both positive).

19: All cross terms are \( x_i^T x_j = 0 \) because symmetric matrices have orthogonal eigenvectors. So positive eigenvalues \( \Rightarrow \) positive energy.

20: (a) The determinant is positive; all \( \lambda > 0 \) (b) All projection matrices except \( I \) are singular
(c) The diagonal entries of \( D \) are its eigenvalues (d) \( A = -I \) has \( \text{det} = +1 \) when \( n \) is even.

22: \( R = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ \sqrt{1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} \\ \sqrt{2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \); \( R = Q \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} Q^T = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \).

26: The Cholesky factors \( C = (L \sqrt{D})^T = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \) and \( C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix} \) have
square roots of the pivots from \( D \). Note again \( C^T C = LDL^T = A \).

33: A product \( AB \) of symmetric positive definite matrices comes into many applications.
The “generalized” eigenvalue problem \( Kx = \lambda Mx \) has \( AB = M^{-1}K \). (often we use \( \text{eig}(K, M) \) without actually inverting \( M \).) All eigenvalues \( \lambda \) are positive:

\[ ABx = \lambda x \text{ gives } (Bx)^T ABx = (Bx)^T \lambda x. \text{ Then } \lambda = x^T B^T ABx/x^T Bx > 0. \]
6.6:

2: $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ is similar to $B = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = M^{-1}AM$ with $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

5: $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ are similar (they all have eigenvalues 1 and 0). $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is by itself and also $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is by itself with eigenvalues 1 and $-1$.

8: Same $\Lambda$ Same $S$ But $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$ have the same line of eigenvectors and the same eigenvalues $\lambda = 0, 0$.

12: If $M^{-1}JM = K$ then $JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} = MK = \begin{bmatrix} 0 & m_{12} & m_{13} & 0 \\ 0 & m_{22} & m_{23} & 0 \\ 0 & m_{32} & m_{33} & 0 \\ 0 & m_{42} & m_{43} & 0 \end{bmatrix}$.

That means $m_{21} = m_{22} = m_{23} = m_{24} = 0$. $M$ is not invertible, $J$ not similar to $K$.

21: $J^2$ has three 1’s down the second superdiagonal, and two independent eigenvectors for $\lambda = 0$. Its 5 by 5 Jordan form is $\begin{bmatrix} J_3 & J_2 \\ J_3 & J_2 \end{bmatrix}$ with $J_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $J_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.