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1 (12 pts.)

(a) - Find the eigenvalues and eigenvectors of $A$.

$$A = \begin{bmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{bmatrix}$$

(b) - Write the vector $v = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of $A$.

- Find the vector $A^{10}v$. 

(c) If you solve $\frac{du}{dt} = -Au$ (notice the minus sign), with $u(0)$ a given vector, then as $t \to \infty$ the solution $u(t)$ will always approach a multiple of a certain vector $w$.

- Find this steady-state vector $w$.  

2 (12 pts.)

Suppose $A$ has rank 1, and $B$ has rank 2 ($A$ and $B$ are both $3 \times 3$ matrices).

(a) - What are the possible ranks of $A + B$?

(b) - Give an example of each possibility you had in (a).
(c) - What are the possible ranks of $AB$?

- Give an example of each possibility.
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3 (12 pts.)

(a) - Find the three pivots and the determinant of $A$.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$
(b) - The rank of $A - I$ is _____, so that $\lambda = _____$ is an eigenvalue.

- The remaining two eigenvalues of $A$ are $\lambda = \underline{\quad}$. 

- These eigenvalues are all $\underline{\quad}$, because $A^T = A$. 
(c) The unit eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 \mathbf{x}_1 \mathbf{x}_1^T + \lambda_2 \mathbf{x}_2 \mathbf{x}_2^T + \lambda_3 \mathbf{x}_3 \mathbf{x}_3^T.$$ 

You may compute the $\mathbf{x}_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$. 

4 (12 pts.)

This problem is about $x + 2y + 2z = 0$, which is the equation of a plane through $0$ in $\mathbb{R}^3$.

(a) - That plane is the nullspace of what matrix $A$?

$A = $

- Find an orthonormal basis for that nullspace (that plane).
(b) That plane is the column space of many matrices \( B \).

- Give two examples of \( B \).

(c) - How would you compute the projection matrix \( P \) onto that plane? (A formula is enough)

- What is the rank of \( P \)?
5 (12 pts.)

Suppose \( \mathbf{v} \) is any unit vector in \( \mathbb{R}^3 \). This question is about the matrix \( H \).

\[
H = I - 2\mathbf{v}\mathbf{v}^T.
\]

(a) - Multiply \( H \) times \( H \) to show that \( H^2 = I \).

(b) - Show that \( H \) passes the tests for being a symmetric matrix and an orthogonal matrix.
(c) - What are the eigenvalues of $H$?

You have enough information to answer for any unit vector $v$, but you can choose one $v$ and compute the $\lambda$'s.
6 (12 pts.)

(a) - Find the closest straight line $y = Ct + D$ to the 5 points:

$$(t, y) = (-2, 0), \ (-1, 0), \ (0, 1), \ (1, 1), \ (2, 1).$$
(b) - The word "closest" means that you minimized which quantity to find your line?

(c) - If $A^T A$ is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated).
7 (12 pts.)

This symmetric Hadamard matrix has orthogonal columns:

$$H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}, \quad \text{and} \quad H^2 = 4I. $$

(a) What is the determinant of $H$?

(b) What are the eigenvalues of $H$? (Use $H^2 = 4I$ and the trace of $H$).
(c) What are the singular values of $H$?
8 (16 pts.)

In this TRUE/FALSE problem, you should circle your answer to each question.

(a) Suppose you have 101 vectors \(v_1, v_2, \ldots, v_{101} \in \mathbb{R}^{100} \).
- Each \(v_i\) is a combination of the other 100 vectors: TRUE – FALSE
- Three of the \(v_i\)'s are in the same 2-dimensional plane: TRUE – FALSE

(b) Suppose a matrix \(A\) has repeated eigenvalues 7, 7, 7, so \(\det(A - \lambda I) = (7 - \lambda)^3\).
- Then \(A\) certainly cannot be diagonalized (\(A = SAS^{-1}\)): TRUE – FALSE
- The Jordan form of \(A\) must be \(J = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix} \): TRUE – FALSE

(c) Suppose \(A\) and \(B\) are \(3 \times 5\).
- Then \(\text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B)\): TRUE – FALSE

(d) Suppose \(A\) and \(B\) are \(4 \times 4\).
- Then \(\det(A + B) \leq \det(A) + \det(B)\): TRUE – FALSE

(e) Suppose \(u\) and \(v\) are orthonormal, and call the vector \(b = 3u + v\). Take \(V\) to be the line of all multiples of \(u + v\).
- The orthogonal projection of \(b\) onto \(V\) is \(2u + 2v\): TRUE – FALSE

(f) Consider the transformation \(T(x) = \int_{-x}^{x} f(t)dt\), for a fixed function \(f\). The input is \(x\), the output is \(T(x)\).
- Then \(T\) is always a linear transformation: TRUE – FALSE
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This is the end of 18.06. Hope you enjoyed learning Linear Algebra!