Your PRINTED name is:  

Please circle your recitation.

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>r01</td>
<td>T 11</td>
<td>4-159</td>
<td>Ailsa Keating</td>
</tr>
<tr>
<td>r02</td>
<td>T 11</td>
<td>36-153</td>
<td>Rune Haugseng</td>
</tr>
<tr>
<td>r03</td>
<td>T 12</td>
<td>4-159</td>
<td>Jennifer Park</td>
</tr>
<tr>
<td>r04</td>
<td>T 12</td>
<td>36-153</td>
<td>Rune Haugseng</td>
</tr>
<tr>
<td>r05</td>
<td>T 1</td>
<td>4-153</td>
<td>Dimiter Ostrev</td>
</tr>
<tr>
<td>r06</td>
<td>T 1</td>
<td>4-159</td>
<td>Uhi Rinn Suh</td>
</tr>
<tr>
<td>r07</td>
<td>T 1</td>
<td>66-144</td>
<td>Ailsa Keating</td>
</tr>
<tr>
<td>r08</td>
<td>T 2</td>
<td>66-144</td>
<td>Niels Martin Moller</td>
</tr>
<tr>
<td>r09</td>
<td>T 2</td>
<td>4-153</td>
<td>Dimiter Ostrev</td>
</tr>
<tr>
<td>r10</td>
<td>ESG</td>
<td></td>
<td>Gabrielle Stoy</td>
</tr>
</tbody>
</table>
(a) - Find the eigenvalues and eigenvectors of $A$.

$$ A = \begin{bmatrix}
3 & 1 & 4 \\
0 & 1 & 5 \\
0 & 1 & 5
\end{bmatrix} $$

*Solution.* The eigenvalues are:

$$ \lambda = 0, 3, 6 $$

The corresponding eigenvectors are:

- $\lambda = 0$ : $v_1 = \begin{bmatrix} 1 \\ -15 \\ 3 \end{bmatrix}$
- $\lambda = 3$ : $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
- $\lambda = 6$ : $v_3 = \begin{bmatrix} 5 \\ 3 \\ 3 \end{bmatrix}$

(b) - Write the vector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of eigenvectors of $A$.

- Find the vector $A^{10} \mathbf{v}$.

*Solution.* We have that, forming $T = [v_1 \mid v_2 \mid v_3]$ (with columns $=$ the three vectors),

$$ \mathbf{y} = T^{-1} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2/3 \\ 1/3 \end{bmatrix} $$
Or in other words:

\[
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix} = 0 \begin{bmatrix}
1 \\
-15 \\
3
\end{bmatrix} - \frac{2}{3} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + \frac{1}{3} \begin{bmatrix}
5 \\
3 \\
3
\end{bmatrix}
\]

Therefore, we also see:

\[
A^{10} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} = -3^{10} \frac{2}{3} \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} + 6^{10} \frac{1}{3} \begin{bmatrix}
5 \\
3 \\
3
\end{bmatrix} \overset{(*)}{=} \begin{bmatrix}
100737594 \\
60466176 \\
60466176
\end{bmatrix}
\]

(*) Required for mental arithmetics wizards only.

(c) If you solve \( \frac{du}{dt} = -Au \) (notice the minus sign), with \( u(0) \) a given vector, then as \( t \to \infty \) the solution \( u(t) \) will always approach a multiple of a certain vector \( w \).

- Find this steady-state vector \( w \).

\textit{Solution.} Since the eigenvalues of \(-A\) are 0, \(-3\), \(-6\), we see that this steady state is:

\[
w = v_1 = \begin{bmatrix}
1 \\
-15 \\
3
\end{bmatrix}
\]
Suppose $A$ has rank 1, and $B$ has rank 2 ($A$ and $B$ are both $3 \times 3$ matrices).

(a) - What are the possible ranks of $A + B$?

*Solution.* Of course, $0 \leq \text{rank}(A + B) \leq 3$. But the only ranks that are possible are:

$$\text{rank}(A + B) = 1, 2, 3.$$  

The reason 0 is not an option is: It implies $A + B = 0$, i.e. that $A = -B$. But $\text{rank}(-B) = \text{rank}(B)$, so for that to happen $A$ and $B$ should have had the same rank.  

(b) - Give an example of each possibility you had in (a).

*Solution.* Here are some simple examples:

**Example w/ $\text{rank}(A + B) = 1$:** Take e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Example w/ $\text{rank}(A + B) = 2$:** Take e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Example w/ $\text{rank}(A + B) = 3$:** Take e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(c) - What are the possible ranks of $AB$?

- Give an example of each possibility.
Solution. As a general rule, recall $0 \leq \text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B)) = 1$. In this case, both possibilities do happen:

$$\text{rank}(AB) = 0, 1.$$ 

Diagonal examples suffice:

**Example w/ rank(AB) = 0:** Take e.g.

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

**Example w/ rank(AB) = 1:** Take e.g.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

□
3 (12 pts.)

(a) - Find the three pivots and the determinant of $A$.

\[
A = \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
-1 & 1 & 0
\end{bmatrix}
\]

**Solution.** We see that

\[
A \sim \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & -2
\end{bmatrix}
\]

Thus,

The pivots are 1, 1, −2

Since we reduced $A$ without any *row switches* (permutation $P$’s), or row scalings, we have:

\[
\det A = 1 \cdot 1 \cdot (-2) = -2
\]

(b) - The rank of $A - I$ is _____, so that $\lambda = _____$ is an eigenvalue.

- The remaining two eigenvalues of $A$ are $\lambda = ____________.

- These eigenvalues are all ____________________, because $A^T = A$.

**Solution.** We see that

\[
\text{rank}(A - I) = 2
\]

So dim $N(A - I) = 1$. Thus,

\[
\lambda = 1
\]

is an eigenvalue of algebraic and geometric multiplicity one.

The other two eigenvalues of $A$ are:

\[
\lambda = -1, 2.
\]
The eigenvalues are all real values, because $A$ is symmetric.

(c) The unit eigenvectors $x_1, x_2, x_3$ will be orthonormal.

- Prove that:

$$A = \lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \lambda_3 x_3 x_3^T.$$ 

You may compute the $x_i$'s and use numbers. Or, without numbers, you may show that the right side has the correct eigenvectors $x_1, x_2, x_3$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3$.

**Solution.** As suggested, we check that $A$ does the correct thing on the basis \{${x_1, x_2, x_2}$\}.

$$(\lambda_1 x_1 x_1^T + \lambda_2 x_2 x_2^T + \lambda_3 x_3 x_3^T) x_i = \lambda_i (x_i^T x_i) x_i = \lambda_i x_i = A x_i$$

Having checked this, then by linearity of matrix multiplication, the two expressions agree always (and hence the matrices are identical).

For the record, the three vectors are:

$$x_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$x_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$x_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$
4 (12 pts.)

This problem is about \( x + 2y + 2z = 0 \), which is the equation of a plane through \( 0 \) in \( \mathbb{R}^3 \).

(a) - That plane is the nullspace of what matrix \( A \)?

\[
A = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}
\]

- Find an orthonormal basis for that nullspace (that plane).

\textit{Solution.}

We could identify a basis of \( N(A) \) as usual, then apply Gram-Schmidt to make it an orthonormal basis.

But if we can find two orthonormal vectors in \( N(A) \), we are done. Here, one can first easily guess one vector in \( N(A) \):

\[
v_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \in N(A)
\]

Then anything of the form \( \begin{bmatrix} a & 1 & 1 \end{bmatrix} \) will be orthogonal to \( v_1 \), and we pick the one that is in the null space:

\[
v_2 = \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix} \in N(A)
\]

Then an orthonormal basis is:

\[
q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \quad q_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -4 \\ 1 \\ 1 \end{bmatrix}
\]

(b) That plane is the column space of many matrices \( B \).

- Give two examples of \( B \).
Solution. We can use the basis vectors from above as columns, and (independent) linear combinations of them. Or filling in a zero column:

\[
B_1 = \begin{bmatrix} v_1 & v_2 \end{bmatrix}
\]

\[
B_2 = \begin{bmatrix} v_1 & 2v_1 + v_2 \end{bmatrix}
\]

\[
B_3 = \begin{bmatrix} v_1 & v_2 & 0 \end{bmatrix}
\]

Then \( c(B_i) = N(A) \).

(c) - How would you compute the projection matrix \( P \) onto that plane? (A formula is enough)

- What is the rank of \( P \)?

Solution. It can be computed using a matrix \( B \) from above (if it has independent columns: So \( B_1, B_2 \) but not \( B_3 \) here), via the usual formula:

\[
P = B(B^TB)^{-1}B^T
\]

For a projection, \( c(P) \) is always the subspace it projects on, in this case it is the two-dimensional plane:

\[
\text{rank}(P) = \text{dim } c(P) = 2
\]
Suppose \( \mathbf{v} \) is any unit vector in \( \mathbb{R}^3 \). This question is about the matrix \( H \).

\[
H = I - 2\mathbf{v}\mathbf{v}^T.
\]

(a) - Multiply \( H \) times \( H \) to show that \( H^2 = I \).

\[
H^2 = (I - 2\mathbf{v}\mathbf{v}^T)^2 = I^2 + 4(\mathbf{v}\mathbf{v}^T)^2 - 4\mathbf{v}\mathbf{v}^T = I + 4\mathbf{v}\mathbf{v}^T - 4\mathbf{v}\mathbf{v}^T = I
\]

\( \square \)

(b) - Show that \( H \) passes the tests for being a symmetric matrix and an orthogonal matrix.

\[
(I - 2\mathbf{v}\mathbf{v}^T)^T = I - 2(\mathbf{v}^T)^Tv^T = I - 2\mathbf{v}\mathbf{v}^T
\]

For orthogonality, we use (a) and symmetry:

\[
HH^T = H^2 = I
\]

\( \square \)

(c) - What are the eigenvalues of \( H \)?

You have enough information to answer for any unit vector \( \mathbf{v} \), but you can choose one \( \mathbf{v} \) and compute the \( \lambda \)'s.

\[
\lambda = -1
\]

Let on the other hand \( \mathbf{u} \in (\text{span}\{\mathbf{v}\})^\perp \) be any vector orthogonal to \( \mathbf{v} \). Then we have:

\[
H\mathbf{u} = \mathbf{u} - 2(\mathbf{v}^T\mathbf{u})\mathbf{v} = \mathbf{u},
\]

so that

\[
\lambda = 1
\]
is also an eigenvalue.

Since \((\text{span}\{v\})^\perp\) is two-dimensional, we have found all eigenvalues.

\[\square\]

6 (12 pts.)

(a) - Find the closest straight line \(y = Ct + D\) to the 5 points:

\((t, y) = (-2, 0), (-1, 0), (0, 1), (1, 1), (2, 1)\).

**Solution.** We insert all points into the equation:

\[-2C + D = 0\]
\[-C + D = 0\]
\[0 + D = 1\]
\[1 + D = 1\]
\[2C + D = 1.\]

Written as a matrix system:

\[
\begin{bmatrix}
-2 & 1 \\
-1 & 1 \\
0 & 1 \\
1 & 1 \\
2 & 1
\end{bmatrix}
\begin{bmatrix}
C \\
D
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{bmatrix}
= \mathbf{b}
\]

We consider instead \(A^T \mathbf{A} \mathbf{x} = A^T \mathbf{b}\). We compute:

\[
A^T A = \begin{bmatrix}
10 & 0 \\
0 & 5
\end{bmatrix}, \quad A^T \mathbf{b} = \begin{bmatrix}
3 \\
3
\end{bmatrix}
\]

and

\[
(A^T A)^{-1} = \begin{bmatrix}
1/10 & 0 \\
0 & 1/5
\end{bmatrix}.
\]

Thus finally:

\[
\begin{bmatrix}
C \\
D
\end{bmatrix}
= \mathbf{x} = (A^T A)^{-1} A^T \mathbf{b}
= \begin{bmatrix}
3/10 \\
3/5
\end{bmatrix}.
\]
So, the closest line to the five points is:

\[ y = \frac{3}{10} t + \frac{3}{5}. \]

(b) - The word "closest" means that you minimized which quantity to find your line?

Solution. It means that the sum of squares deviation \( \|Ax - b\|^2 \) was minimized.

(c) If \( A^T A \) is invertible, what do you know about its eigenvalues and eigenvectors? (Technical point: Assume that the eigenvalues are distinct – no eigenvalues are repeated). Since \( A^T A \) is symmetric and \( x \cdot (A^T A x) = \|Ax\|^2 \geq 0 \) always, it is positive semi-definite. Since \( N(A^T A) = \{0\} \), zero is not eigenvalue. Hence:

The eigenvalues of \( A^T A \) are positive, if \( A^T \) is invertible

By symmetry:

Eigenvectors belonging to different eigenvalues are orthogonal
This symmetric Hadamard matrix has orthogonal columns:

\[ H = \begin{bmatrix}
1 & 1 & 1 & 1 \\
1 & -1 & 1 & -1 \\
1 & 1 & -1 & -1 \\
1 & -1 & -1 & 1
\end{bmatrix}, \quad \text{and} \quad H^2 = 4I. \]

(a) What is the determinant of \( H \)?

\[ \text{Solution. By row reduction, we get the pivots 1, -2, -2, 4, so:} \]

\[ \det H = 16 \]

\[ \square \]

(b) What are the eigenvalues of \( H \)? (Use \( H^2 = 4I \) and the trace of \( H \)).

\[ \text{Solution. By } H^2 = 4I, \text{ the eigenvalues are all either } \pm 2. \text{ They sum up to } \text{tr}H = 0. \text{ Hence:} \]

\[ \text{Two eigenvalues must be } +2, \text{ and two eigenvalues be } -2 \]

\[ \text{Note also that this shows } \det H = 16 \text{ as in } (a) \]

\[ \square \]

(c) What are the singular values of \( H \)?

\[ \text{The singular values of } H \text{ are } 2, 2, 2, 2 \]
In this TRUE/FALSE problem, you should circle your answer to each question.

(a) Suppose you have 101 vectors \( v_1, v_2, \ldots, v_{101} \in \mathbb{R}^{100} \).

- Each \( v_i \) is a combination of the other 100 vectors: \( \text{TRUE} - \text{FALSE} \)

- Three of the \( v_i \)'s are in the same 2-dimensional plane: \( \text{TRUE} - \text{FALSE} \)

(b) Suppose a matrix \( A \) has repeated eigenvalues 7, 7, 7, so \( \det(A - \lambda I) = (7 - \lambda)^3 \).

- Then \( A \) certainly cannot be diagonalized (\( A = SAS^{-1} \)): \( \text{TRUE} - \text{FALSE} \)

- The Jordan form of \( A \) must be \( J = \begin{bmatrix} 7 & 1 & 0 \\ 0 & 7 & 1 \\ 0 & 0 & 7 \end{bmatrix} \): \( \text{TRUE} - \text{FALSE} \)

(c) Suppose \( A \) and \( B \) are \( 3 \times 5 \).

- Then \( \text{rank}(A + B) \leq \text{rank}(A) + \text{rank}(B) \): \( \text{TRUE} - \text{FALSE} \)

(d) Suppose \( A \) and \( B \) are \( 4 \times 4 \).

- Then \( \det(A + B) \leq \det(A) + \det(B) \): \( \text{TRUE} - \text{FALSE} \)

(e) Suppose \( u \) and \( v \) are orthonormal, and call the vector \( b = 3u + v \). Take \( V \) to be the line of all multiples of \( u + v \).

- The orthogonal projection of \( b \) onto \( V \) is \( 2u + 2v \): \( \text{TRUE} - \text{FALSE} \)

(f) Consider the transformation \( T(x) = \int_{-x}^{x} f(t) dt \), for a fixed function \( f \). The input is \( x \), the output is \( T(x) \).

- Then \( T \) is always a linear transformation: \( \text{TRUE} - \text{FALSE} \)