18.06 Spring 2012 – Problem Set 1

This problem set is due Thursday, February 16th, 2012 at 4pm (hand in to Room 2-106). The textbook problems are out of the 4th edition. For computational problems, please include a printout of the code with the problem set (for MATLAB in particular, `diary('filename')` will start a transcript session, `diary off` will end one.)

Every problem is worth 10 points.

1. Do Problem 8 from Section 1.3.

2. Do Problem 8 & Problem 32 from Section 2.2.

3. Do Problem 22 from Section 2.3.

4. Do Problem 19 & Problem 36 from Section 2.4.

5. For which values of $q$ (if any) is the following system consistent (= solvable)?

$$
\begin{align*}
x + 4y + 3z &= 1, \\
q^3 x + 4q^3 y + 3q^3 z &= 64q.
\end{align*}
$$

6. A permutation matrix $P$ comes from permuting the rows of the identity matrix $I_n$. If the entries of $P$ are labelled $p_{ij}$, the matrix $A$ having entries $a_{ij} = p_{ji}$ is the transpose, $A = P^T$.

(a) Is $P$ invertible, and if yes why? How would we proceed in Gaussian elimination on $P$?

(b) Explain why the product $C = PP^T$ is the identity matrix. Think about where the 1’s and 0’s are.

(c) Since the answer to (a) was "yes", what is the inverse to $P$?

7. (a) Give examples of non-zero (meaning: not all entries zero) $2 \times 2$ and $4 \times 4$ matrices $A$, one of each, such that $A^2 = O$ (recall $O$ means the zero matrix). Hint: You only need to use one 1, and the rest of the entries can be 0’s!

(b) Are there any invertible $n \times n$ matrices $A$ such that $A^2 = O$?

8. Given the three vectors $a_1 = (1, 2, 3)$, $a_2 = (1, 0, -1)$ and $a_3 = (0, 0, 1)$, find (if possible) numbers $x_1, x_2$ and $x_3$ such that:

$$
x_1a_1 + x_2a_2 + x_3a_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.
$$

Your solution should involve Gaussian elimination on $A = [a_1 \ a_2 \ a_3]$ (the matrix with $a_i$'s as columns).
9. (a) Using MATLAB, perform the matrix products $A^2$, $A^3$ and $A^6$ of the following lower-triangular matrix:

\[
A = \begin{bmatrix}
1 & 0 & 0 & 0 \\
7 & 2 & 0 & 0 \\
5 & 1 & 3 & 0 \\
3 & 2 & -1 & 4
\end{bmatrix}
\]

(b) Explain the rule for diagonal entries of $A^k$, for a lower-triangular matrix $A$.

(c) Guess a rule for the $(2, 1)$ entry of $A^k$, for a lower-triangular matrix $A$.

10. A chemistry professor claimed on live TV that he could, by mixing, obtain any wine with given contents of water (W), sugar (S) and tannic acid (T), labelled by vectors $w = (W, S, T)$ such that $W + S + T = 100\%$. Due to a lack of research funding, his stock was quite limited:

- Laboratory water supply: $w_1 = (100, 0, 0)$.
- Budget wine: $w_2 = (50, 0, 50)$.
- Plum tea concentrate: $w_3 = (30, 50, 20)$.

(a) If a Chateaux Bordeaux 1915 has $(W, S, T) = (45, 50, 5)$, why was the professor not able to obtain this wine by mixing $w_1, w_2, w_3$? Explain by computing the mixing ratios needed (by MATLAB or by hand).

(b) Help the professor restore honor, by adding any new wine $w_4$ that will enable him to make the Chateaux Bordeaux 1915 (a Chateaux Bordeaux 1915 not allowed!).

(c) Are the mixing ratios unique after addition of the fourth wine?