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1 (33 pts.)

Suppose an \( n \times n \) matrix \( A \) has \( n \) independent eigenvectors \( x_1, \ldots, x_n. \) Then you could write the solution to \( \frac{du}{dt} = Au \) in three ways:

\[
u(t) = e^{At}u(0), \quad \text{or} \quad u(t) = Se^{\Lambda t}S^{-1}u(0), \quad \text{or} \quad u(t) = c_1e^{\lambda_1t}x_1 + \ldots + c_ne^{\lambda_nt}x_n.
\]

Here, \( S = [x_1 \mid x_2 \mid \ldots \mid x_n]. \)

(a) From the definition of the exponential of a matrix, show why \( e^{At} \) is the same as \( Se^{\Lambda t}S^{-1}. \)

\text{Solution.} \text{ Recall that } A = SAS^{-1}, \text{ and } A^k = SA^kS^{-1}. \text{ Then, definition of the exponential:}

\[
\exp(At) = \sum_{k=0}^{\infty} \frac{A^k}{k!} = S \left( \sum_{k=0}^{\infty} \frac{\Lambda^k}{k!} \right) S^{-1} = Se^{\Lambda t}S^{-1}.
\]

(b) How do you find \( c_1, \ldots, c_n \) from \( u(0) \) and \( S? \)

\text{Solution.} \text{ Since } e^0 = 1, \text{ we see that}

\[
u(0) = c_1x_1 + \ldots + c_nx_n = S\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix},
\]

where we used the definition of the matrix product. Thus the answer is:

\[
\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = S^{-1}u(0).
\]
(c) For this specific equation, write $u(t)$ in any one of the (added: latter two of the) three forms, using *numbers* not symbols: You can choose which form.

$$\frac{du}{dt} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} u, \quad \text{starting from} \quad u(0) = \begin{bmatrix} 4 \\ 3 \end{bmatrix}.$$ 

*Solution.* We diagonalize $A$ and get:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}.$$ 

Thus $c = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, so for the second form

$$u(t) = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \left( \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right),$$

while in the third form:

$$u(t) = e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 2e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$ 

$\square$
This question is about the real matrix

\[ A = \begin{bmatrix} 1 & c \\ 1 & -1 \end{bmatrix}, \quad \text{for} \; \ c \in \mathbb{R}. \]

(a) - Find the eigenvalues of \( A \), depending on \( c \).

- For which values of \( c \) does \( A \) have real eigenvalues?

*Solution.* Since \( 0 = \text{tr}A = \lambda_1 + \lambda_2 \), we see that \( \lambda_2 = -\lambda_1 \).

Also, \(-1 - c = \det A = -\lambda_1^2 \). Thus,

\[
\lambda = \pm \sqrt{1 + c}.
\]

Therefore,

*the eigenvalues are real precisely when \( c \geq -1 \).*
(b) - For one particular value of $c$, convince me that $A$ is similar to both the matrix

$$B = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix},$$

and to the matrix

$$C = \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}.$$  

- Don’t forget to say which value $c$ this happens for.

**Solution.** If two matrices are similar, then they do have the same eigenvalues (those are $2, -2$ for both $B$ and $C$). Here we must therefore have $0 = \text{tr} A$ and $-1 - c = \det A = -4$. We see that this happens precisely when $c = 3$, where we check that indeed the eigenvalues are $2, -2$. However, this does not guarantee that they are similar - and hence is not convincing.

Convincing: The eigenvalues $2, -2$ are different, so both $A$, $B$ and $C$ are diagonalizable, with the same diagonal matrix (for example to $\Lambda = B$!). Therefore $A$, $B$ and $C$ are all similar when $c = 3$. \qed
(c) For one particular value of $c$, convince me that $A$ cannot be diagonalized. It is not similar to a diagonal matrix $\Lambda$, when $c$ has that value.

- Which value $c$?
- Why not?

*Solution.* As we saw above, $\text{tr} A = 0$, so regardless of $c$ the eigenvalues come in pairs $\lambda_2 = -\lambda_1$. This means that whenever $\lambda_1 \neq 0$, we have two different eigenvalues, and hence $A$ is diagonalizable (not what we’re after).

Thus we need $\lambda_1 = \lambda_2 = 0$, a repeated eigenvalue, which happens when $c = -1$ (so $\det A = 0$) as the only suspect – does it work?

\[
\text{Convincing: For } c = -1, \text{ we have } N(A - 0 \cdot I) = \text{span } \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

With only a 1-dimensional space of eigenvectors for the matrix, we are convinced that $A$ is not diagonalizable for $c = -1$. □
3 (37 pts.)

(a) Suppose $A$ is an $n \times n$ symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$.

- What is the largest number real number $c$ that can be subtracted from the diagonal entries of $A$, so that $A - cI$ is positive semidefinite?

- Why?

Solution. - We first realize that: If $A$ is symmetric, then $A - cI$ is also symmetric, since in general $(A + B)^T = A^T + B^T$ (simple, but very important to check!).

- Then we realize that the eigenvalues of $A - cI$ are $\lambda_1 - c \leq \lambda_2 - c \leq \ldots \leq \lambda_n - c$. Therefore:

$$c = \lambda_1$$

is the largest that can ensure positive semidefiniteness (and it does).
(b) Suppose $B$ is a matrix with independent columns.
- What is the nullspace $N(B)$?
- Show that $A = B^T B$ is positive definite. Start by saying what that means about $x^T Ax$.

**Solution.** - Then $Bx = 0$ only has the zero solution, so $N(B) = \{0\}$.

- Again, we start by observing that $A^T = A$ is symmetric. Then we recall what positive definite means (the "energy" test):

$$x^T Ax > 0 \quad \text{whenever} \quad x \neq 0.$$  

Thus, we see here (by definition the inner product property of the transpose of a matrix):

$$x^T Ax = x^T (B^T (Bx)) = (Bx)^T (Bx) = \|Bx\|^2 \geq 0.$$  

So $A = B^T B$ is positive semidefinite. But finally, the equality $\|Bx\|^2 = 0$, only happens when $Bx = 0$ which by $N(B) = \{0\}$ means $x = 0$.  \[\square\]
(c) This matrix $A$ has rank $r = 1$:

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$  

- Find its largest singular value $\sigma$ from $A^T A$.

- From its column space and row space, respectively, find unit vectors $u$ and $v$ so that

$$Av = \sigma u, \quad \text{and} \quad A = u \sigma v^T.$$  

- From the nullspaces of $A$ and $A^T$ put numbers into the full SVD (Singular Value Decomposition) of $A$:

$$A = \begin{bmatrix} | & | \\ u & \cdots \\ | & | \end{bmatrix} \begin{bmatrix} \sigma & 0 \\ 0 & \ddots \end{bmatrix} \begin{bmatrix} | & | \\ v & \cdots \\ | & | \end{bmatrix}^T.$$  

**Solution.** We compute:

$$A^T A = \begin{bmatrix} 5 & 5 \\ 5 & 5 \end{bmatrix}.$$  

Thus the two eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 10$, and $\sigma = \sqrt{10}$. For $v$, we find a vector in $N(A^T A - 10 I)$, and normalize to unit length:

$$v = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}.$$  

Then we find $u$ using

$$u = \frac{Av}{\sigma} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}.$$  

Since we have the orthogonal sums of subspaces $\mathbb{R}^2 = \mathbb{R}^m = c(A) \oplus N(A^T)$ and also $\mathbb{R}^2 = \mathbb{R}^n = c(A^T) \oplus N(A)$, we need to find one unit vector from each of $N(A)$ and $N(A^T)$ and augment to $v$ and $u$, respectively:

$$v_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \in N(A),$$  

$$u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \in N(A^T).$$
Thus, we finally see the full SVD:

\[
A = U\Sigma V^T = \begin{bmatrix}
\frac{1}{\sqrt{5}} & -2\sqrt{\frac{2}{5}} \\
2\sqrt{\frac{2}{5}} & \frac{1}{\sqrt{5}}
\end{bmatrix}
\begin{bmatrix}
\sqrt{10} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\frac{1}{\sqrt{2}} & 1/\sqrt{2} \\
1/\sqrt{2} & -1/\sqrt{2}
\end{bmatrix}^T.
\]

We remember, as a final check, to verify that the square matrices \(U\) and \(V\) both contain orthonormal bases of \(\mathbb{R}^2\) as they should:

\[
UU^T = I_2,
\]

\[
VV^T = I_2.
\]