Problem 1 (30 pts)
(a) Since the multipliers are all 3, the row operations we had goes:
– subtract three times row 1 to row 2;
– subtract three times row 1 to row 3;
– subtract three times row 2 to row 3,
where the end step gives us
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 1 \\
0 & 0 & g
\end{bmatrix}.
\]
So now we just need to reverse the row operations. Reversing the last step means adding three times row 2 to row 3:
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 1 \\
0 & 6 & g + 3
\end{bmatrix}.
\]
Reversing the second step means adding three times row 1 to row 3:
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 1 \\
3 & 9 & g + 6
\end{bmatrix},
\]
And reversing the first step means adding three times row 1 to row 2:
\[
\begin{bmatrix}
1 & 1 & 1 \\
3 & 5 & 4 \\
3 & 9 & g + 6
\end{bmatrix}.
\]
(b) To find the nullspace of \( A \), we first note that there are three columns in \( A \), and that the rank of \( A \) is 2 in the case when \( g = 2 \), so the dimension of \( N(A) \) is one. So we need to find just one vector that are in \( N(A) \); then this vector will be a basis for \( N(A) \). By using Gaussian elimination to \( Ax = 0 \), we get \( Ux = 0 \), from which we deduce that \( \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \) is a solution.
Therefore, \( N(A) = c \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \).
(c) If \( g \neq 0 \), then we observe that \( C(U) = \mathbb{R}^3 \). Since row rank is equal to the column rank, the column rank (i.e. the dimension of the column space) of \( A \) is also 3. Since the column space of \( A \) is contained in \( \mathbb{R}^3 \), which is 3-dimensional, it must be the whole space as well. So \( C(A) = \mathbb{R}^3 \).

Problem 2 (40 pts)
(a) Any row of $A$ is orthogonal to the two special solutions given in the problem. That is, any row

$$r = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

satisfies $r \cdot s_1 = r \cdot s_2 = 0$. This is just a system of two linear equations, so we need to solve the equation

$$\begin{bmatrix} 3 & 1 & 0 & 0 \\ 6 & 0 & 2 & 1 \end{bmatrix} r = 0$$

whose complete solution is given by

$$c \begin{bmatrix} 1 \\ -3 \\ 0 \\ -6 \end{bmatrix} + d \begin{bmatrix} 0 \\ 0 \\ -1 \\ 2 \end{bmatrix},$$

from which we get the reduced row echelon form of $A$ given by

$$\begin{bmatrix} 1 & -3 & 0 & -6 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

(b) $R$ has two pivots, and therefore $A$ has two pivots and $r(A) = 2$. Two independent columns in $\mathbb{R}^2$ span $\mathbb{R}^2$, so $C(A) = \mathbb{R}^2$.

Partial credit was given if the student referred back to $A$ for the column space and if they gave $\mathbb{R}^2$ with incomplete reasoning. Most point were lost if they just gave $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as the basis without indicating where they came from (or from reading them off of $R$, not $A$). There were lots of right answer, with wrong (or no) reasons.

(c) The free variables are $x_2, x_4$ so the particular solution is

$$x_p = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix}.$$

The complete solution is

$$x_c = x_p + c_1 \cdot s_1 + c_2 \cdot s_2 = \begin{bmatrix} 3 \\ 0 \\ 6 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 6 \\ 0 \\ 2 \\ 1 \end{bmatrix}.$$

(d) We have

$$2 \cdot \begin{bmatrix} -3 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -6 \\ -2 \end{bmatrix}.$$
One can find these coefficients by inspection, or combining the special solutions:

\[ 2s_1 - s_2 = \begin{bmatrix} 0 \\ 2 \\ -2 \\ -1 \end{bmatrix}. \]

**Problem 3 (30 pts)**

(a) Let \( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \) be a column of \( X \). Then \( x, y \) and \( z \) satisfy

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} = 0.
\]

We apply elimination to get

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} = 0
\]

from which we deduce that \( y = 0 \), and \( x = -z \). So each column of \( X \) is a multiple of the vector \( \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \). Since there are two columns of \( X \), \( X \) can be written as

\[
\begin{bmatrix}
a & b \\ 0 & 0 \\ -a & -b
\end{bmatrix}.
\]

The basis for this space of matrices is given by

\[
\begin{bmatrix}
1 & 0 \\ 0 & 0 \\ -1 & 0
\end{bmatrix},
\begin{bmatrix}
0 & 1 \\ 0 & 0 \\ 0 & -1
\end{bmatrix}.
\]

(b) We first solve

\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.
\]

Elimination gives We apply elimination to get

\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\ y \\ z
\end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

From which we see that we can take \( y = -1/2, x = 3/2, z = 0 \). This will be the first column of \( X \).
We now solve
\[
\begin{bmatrix}
1 & 1 & 1 \\
1 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

Elimination gives
\[
\begin{bmatrix}
1 & 1 & 1 \\
0 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
=
\begin{bmatrix}
0 \\
1
\end{bmatrix}.
\]

And we can take \( y = 1/2, x = -1/2, z = 0 \). This is the second column of \( X \).

So one possible solution for \( X \) is
\[
\begin{bmatrix}
3/2 & -1/2 \\
-1/2 & 1/2 \\
0 & 0
\end{bmatrix}.
\]

(c) The set of complete solutions is given by
\[
X_{\text{particular}} + X_{\text{special}} =
\begin{bmatrix}
3/2 & -1/2 \\
-1/2 & 1/2 \\
0 & 0
\end{bmatrix} + a
\begin{bmatrix}
1 & 0 \\
0 & 0 \\
-1 & 0
\end{bmatrix} + b
\begin{bmatrix}
0 & 1 \\
0 & 0 \\
0 & -1
\end{bmatrix}.
\]