Solutions to Exam 2

Problem 1 (30 pts)
(a) The rank of $P$ is 2. Any vector perpendicular to the subspace spanned by $a_1$ and $a_2$ is in the nullspace of $P$, and the orthogonal complement of the subspace spanned by $a_1$ and $a_2$ is 3-dimensional (that is, there are three independent vectors that project to 0 by $P$). This is exactly the nullspace of $P$, and since

\[ \text{rank } P = \dim \mathcal{C}(P) = 5 - \dim \text{Nullspace } P, \]

the rank of $P$ is $5 - 3 = 2$.

(b) The nullspace of $P$ is the left nullspace of $A$. Indeed, we have

\[ Pv = 0 \iff a_1^T v = 0 \text{ and } a_2^T v = 0 \]

\[ \iff v^T a_1 = 0 \text{ and } v^T a_2 = 0 \]

\[ \iff vA = 0. \]

(c) Gram-Schmidt gives

\[ q_1 = \frac{a_1}{\|a_1\|} = \frac{(1, 0, 1, 0, 4)^T}{\sqrt{1^2 + 0^2 + 1^2 + 0^2 + 4^2}} = \frac{1}{3\sqrt{2}}(1, 0, 1, 0, 4)^T \]

and

\[ q_2 = \frac{a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1}{\|a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1\|} = \frac{a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1}{\|a_2 - \frac{a_2^T q_1}{q_1^T q_1} q_1\|} = \frac{(2, 0, 0, 0, 4)^T - (1, 0, 1, 0, 4)^T}{\|(2, 0, 0, 0, 4)^T - (1, 0, 1, 0, 4)^T\|} \]

\[ = \frac{1}{\sqrt{2}}(1, 0, -1, 0, 0)^T, \]

and $q_1$ and $q_2$ form an orthonormal basis for the column space of $A$.

(d) Since $P$ is a projection matrix, we have $P = P^T$. To show that $Q$ is an orthogonal matrix, we need to check that $QQ^T = I$. We have

\[ QQ^T = (I - 2P)(I - 2P)^T \]

\[ = (I - 2P)(I^T - 2P^T) \]

\[ = (I - 2P)(I - 2P)(I \text{ and } P \text{ are symmetric}) \]

\[ = I - 4P + 4P^2 \]

Since for a projection matrix we have $P^2 = P$, this product is equal to $QQ^T = I$, as required.

Problem 2 (30 pts)
(a) We will find the determinant by doing row operations:

\[
\begin{vmatrix}
1 & 2 & 0 & 0 \\
1 & 2 & 3 & 0 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 4
\end{vmatrix}
= \det
\begin{vmatrix}
1 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 4
\end{vmatrix}
= -\det
\begin{vmatrix}
1 & 2 & 0 & 0 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{vmatrix}
= -\det
\begin{vmatrix}
1 & 2 & 0 & 0 \\
0 & 2 & 3 & 4 \\
0 & 0 & 3 & 0 \\
0 & 0 & 0 & 4
\end{vmatrix}
\]

and the last matrix has determinant \((1) \cdot (2) \cdot (3) \cdot (4) = 24\), so the original matrix has determinant \(-24\).

(b) \(\det A\) tells the volume of a box in \(\mathbb{R}^4\) whose sides are given by the vectors \((1, 1, 0, 0)^T\), \((2, 2, 2, 0)^T\), \((0, 3, 3, 3)^T\), and \((0, 0, 4, 4)^T\). Another box with the same volume would be a box whose sides are given by the vectors \((1, 0, 0, 0)^T\), \((2, 2, 0, 0)^T\), \((0, 3, 3, 0)^T\), and \((0, 4, 0, 4)^T\).

(These are obtained from \(A\) via row operations, and so the absolute value of the determinants do not change!)

(c) The formula for \(A^{-1}\) says that (see page 270 of the textbook!)

\[(A^{-1})_{ij} = \frac{C_{ji}}{-\det A}\]

where \(C_{ji}\) is the cofactor given by removing row \(j\) and column \(i\). From the problem, this matrix is not invertible, so its determinant is 0, meaning that \(C_{ij} = 0\). This means that the \((4, 3)\)-entry of \(A^{-1}\) is also 0.

**Problem 3 (30 pts)**

(a) Letting

\[A = \begin{bmatrix}
1 \\
-1 \\
-1 \\
2
\end{bmatrix},\]
the projection matrix that projects every \( b \in R^4 \) onto the column space of \( A \) (which is the line through \( q_4 \)) is given by the formula

\[
A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}^{-1} 
\]

\[
= \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} 
\]

(b) Letting

\[
A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix} 
\]

the projection matrix that projects every \( b \in R^4 \) onto the column space of \( A \) (which is the subspace spanned by \( q_1, q_2 \) and \( q_3 \)) is given by the formula

\[
A(A^T A)^{-1} A^T = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}^{-1} 
\]

\[
= \frac{1}{4} \begin{bmatrix} 3 & 1 & 1 & -1 \\ 1 & 3 & -1 & 1 \\ 1 & -1 & 3 & 1 \\ -1 & 1 & 1 & 3 \end{bmatrix} 
\]

(c) We must solve the new system

\[
A^T \hat{x} = A^T b. 
\]

Since \( A^T A = I \), we have

\[
\hat{x} = A^T b = \begin{bmatrix} 5 \\ -1 \\ -2 \end{bmatrix}.
\]

Then \( A\hat{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \), and \( e = b - A\hat{x} = 0. \)