(1) (40 pts)

In all of this problem, the 3 by 3 matrix $A$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with independent eigenvectors $x_1, x_2, x_3$.

(a) What are the trace of $A$ and the determinant of $A$?

(b) Suppose: $\lambda_1 = \lambda_2$. Choose the true statement from 1, 2, 3:

1. $A$ can be diagonalized. Why?
2. $A$ can not be diagonalized. Why?
3. I need more information to decide. Why?

(c) From the eigenvalues and eigenvectors, how could you find the matrix $A$? Give a formula for $A$ and explain each part carefully.

(d) Suppose $\lambda_1 = 2$ and $\lambda_2 = 5$ and $x_1 = (1, 1, 1)$ and $x_2 = (1, -2, 1)$. Choose $\lambda_3$ and $x_3$ so that $A$ is symmetric positive semidefinite but not positive definite.
(2) (30 pts.)

Suppose $A$ has eigenvalues $1, \frac{1}{3}, \frac{1}{2}$ and its eigenvectors are the columns of $S$:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{with} \quad S^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

(a) What are the eigenvalues and eigenvectors of $A^{-1}$?

(b) What is the general solution (with 3 arbitrary constants $c_1, c_2, c_3$) to the differential equation $du/dt = Au$? Not enough to write $e^{At}$. Use the $c$’s.

(c) Start with the vector $u = (1, 4, 3)$ from adding up the three eigenvectors:

$u = x_1 + x_2 + x_3$. Think about the vector $v = A^k u$ for VERY large powers $k$.

What is the limit of $v$ as $k \to \infty$?
(3) (30 pts.)

(a) For a really large number \( N \), will this matrix be positive definite? Show why or why not.

\[
A = \begin{bmatrix}
2 & 4 & 3 \\
4 & N & 1 \\
3 & 1 & 4
\end{bmatrix}.
\]

(b) Suppose: \( A \) is positive definite symmetric
\( Q \) is orthogonal (same size as \( A \))
\( B \) is \( Q^T AQ = Q^{-1} AQ \)

Show that: 1. \( B \) is also symmetric.
2. \( B \) is also positive definite.

(c) If the SVD of \( A \) is \( U \Sigma V^T \), how do you find the orthogonal \( V \) and the diagonal \( \Sigma \) from the matrix \( A \)?